Some remarks on D-branes in AdS_3

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Abstract

We investigate the relation between the algebraic construction of boundary states in AdS_3 and the target space analysis of D-branes and show the consistency of the two descriptions. We compute, in the semiclassical regime, the overlap of a localized closed string state with boundary states and identify the latter with D-branes wrapping conjugacy classes in AdS_3. The string partition function on the disk is shown to reproduce the spacetime DBI action. Other consistency checks are performed. We also comment on the role of the spectral flow symmetry of the underlying SL(2,R)/U(1) coset model in constructing D-branes that correspond to degenerate representations of SL(2,R).

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1 Introduction

Understanding D-branes in AdS_3 is an interesting problem, which received much attention recently [1–8]. The main difficulty stems from the non-compactness of the space, which affects both the algebraic construction of boundary states and the target space analysis. The algebraic approach based on the modular transformation of characters of the WZNW model [9] is a powerful technique for identifying possible D-branes. However, the application of this technique has been so far limited to rational CFTs. In contrast, the SL(2,R) WZNW model, which describes bosonic string theory on AdS_3, is not rational. The alternative target space analysis of D-branes on group manifolds has been developed quite recently [10–13]. One of the important results of this analysis was the understanding that maximally symmetric D-branes, which preserve half of the current algebra, appear as surfaces wrapping conjugacy classes in group manifolds. In the SU(2) case, where conjugacy classes are two-spheres, this result was shown to be consistent with the algebraic description [14, 15]. In fact, the spectrum of the theory on the brane, obtained by expanding the DBI action to the second order in fluctuating fields, was shown to exactly match the CFT result [16]. In the AdS_3 case important conjugacy classes are dS_2, H_2 and AdS_2 surfaces. The target space analysis allows computation of the DBI action for corresponding branes [2]. The algebraic description of D-branes in AdS_3 was pursued in [1], who used conformal bootstrap in CFT on the disk (or the upper half plane) to determine allowed boundary states. More specifically, the crossing symmetry of the two-point function on the disk and the properties of degenerate operators in the SL(2,R) WZNW model have been used to derive the constraint equation for the one-point function of an arbitrary primary of the current algebra. By solving the constraint equation one may find all possible boundary states. In this note we try to understand these boundary states better. Our main interest will be in their geometric interpretation, but we also provide a review of conformal bootstrap on the disk and re-derive some results of [1] by different means.

This paper is organized in the following way. Section 2 serves as a review of CFT on AdS_3 and also summarizes previous results on the geometry and the properties of extended D-branes in this background from the target space point of view. This section also explains our notations. Section 3 is devoted to the CFT analysis of D-branes. Here we review the conformal bootstrap of [1] and discuss D-branes that have a finite spectrum of open strings living on them. It was shown in [1] that such branes correspond to finite dimensional representations of SL(2,R), which are labeled by a discrete papameter. We show that this result can be viewed as a consequence of the spectral flow symmetry of the underlying SL(2,R)/U(1) coset. We next review the construction of boundary states that have a continuous spectrum of open strings living on them. These boundary states are labeled by a complex parameter. When this parameter is pure imaginary (pure real), the modular bootstrap provides a correspondence between the D-brane and the principal continuous (principal discrete) representation of SL(2,R). In section 4, which is the main part of this paper, we provide the geometric interpretation of boundary states that possess a continuous spectrum of open strings. By considering an overlap of a closed string state localized in the target space with CFT boundary states, we recover D-branes that appear as surfaces of constant curvature in Euclidean AdS_3[1]. We show that D-branes that correspond to principal continuous representations, wrap two-spheres in H_3^+. We also argue that after analytic continuation to Lorenzian AdS_3, the boundary states that correspond to principal continuous and principal discrete representations give rise to D-branes that wrap dS_2 and H_2 conjugacy classes, respectively. H_2 branes in H_3^+, which become surfaces wrapping AdS_2 conjugacy classes in Lorenzian AdS_3, correspond to boundary states that are labeled by the parameter that is neither real nor imaginary. We comment on the modular bootstrap interpretation of these states. We provide support for the above picture by computing the string partition sum on the disk, and comparing it with the DBI action of corresponding D-branes. Both quantities are divergent in the case of a D-brane that wraps an H_2 surface in H_3^+, since the area of the hyperbolic plane is infinite. We explain how this divergence can be regularized, and show that once this is done, the CFT partition sum reproduces the DBI action up to a finite normalization factor. In section 5 we discuss our results. The solutions of the Knizhnik-Zamolodchikov equations used in the main text appear in Appendix A. Appendix B studies the transformation of conformal blocks under the action of the spectral flow. Appendices C and D contain some useful formulae.

2 Strings and branes in AdS_3

The SL(2,R) group element can be written as

$$g = \begin{pmatrix} X^0 + X^1 & X^2 - X^3 \\ X^2 + X^3 & X^0 - X^1 \end{pmatrix}, \tag{2.1}$$

where

$$(X^0)^2 - (X^1)^2 - (X^2)^2 + (X^3)^2 = 1. \tag{2.2}$$

[1]The Euclidean version of AdS_3 is the hyperbolic space, H_3^+.

One can think about SL(2,R) as a hyperboloid defined by (2.2) in the four-dimensional space with the flat metric $ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 - (dX^3)^2$. One convenient parameterization of SL(2,R) is

$$X^0 + iX^3 = \cosh\rho e^{it}, \qquad X^1 + iX^2 = -\sinh\rho e^{-i\theta}. \tag{2.3}$$

The range of coordinates is $0 \le \rho \le \infty$, $0 \le \theta \le 2\pi$, $0 \le t \le 2\pi$, and the metric is given by $ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\theta^2$. The unit radius AdS_3, which is the universal cover of SL(2,R) is obtained by decompactifying the timelike direction t. We can perform the Wick rotation $X^3 = i\tilde{X}^3$ which gives the hyperbolic space H_3^+. This is a subspace of four-dimensional Minkowski space with the timelike coordinate X^0, parameterized by the equation

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (\tilde{X}^3)^2 = 1, \qquad X^0 \ge 0. \tag{2.4}$$

We will identify the point $g \in H_3^+$ with the complexified matrix (2.1). A useful parameterization that we will need in the following is

$$g = \begin{pmatrix} \gamma\bar{\gamma}e^{\phi} + e^{-\phi} & -\gamma e^{\phi} \\ -\bar{\gamma}e^{\phi} & e^{\phi} \end{pmatrix}. \tag{2.5}$$

γ and $\bar{\gamma}$ are the coordinates on the complex plane (the sphere) while ϕ is the radial coordinate. The boundary of H_3^+ is at $\phi \to \infty$. The metric takes the form $ds^2 = d\phi^2 + e^{2\phi}d\gamma d\bar{\gamma}$. An important class of functions on H_3^+ are those that transform as spin $j \equiv h-1$ representation of SL(2,R):

$$\tilde{\Phi}_h(y,\bar{y}|g) = \frac{1-2h}{\pi}\left[\begin{pmatrix} 1 & y \end{pmatrix} g \begin{pmatrix} 1 \\ \bar{y} \end{pmatrix}\right]^{-2h} = \frac{1-2h}{\pi}\left(\frac{1}{|\gamma - y|^2 e^{\phi} + e^{-\phi}}\right)^{2h}. \tag{2.6}$$

Here y and $\bar{y}$ are the coordinates on the complex plane which parameterize the function. They also appear as the coordinates in the spacetime CFT via AdS/CFT correspondence [17, 18][1]. The introduction of $y, \bar{y}$ [19] is very convenient from the technical point of view. In particular, Knizhnik-Zamolodchikov equations for four-point functions containing degenerate operators become differential equations that can be solved to obtain structure constants [22] or boundary states [1].

Bosonic string theory on AdS_3 is described by the SL(2,R) WZNW model. It will be convenient to consider instead its Euclidean counterpart, the H_3^+ WZNW model. All our

[1] The spacetime coordinates are denoted by $y, \bar{y}$, not by $x, \bar{x}$ as in [17, 18]. This choice of notation was made to avoid confusion with X^i which parameterize AdS_3.

CFT results are therefore pertinent to the Euclidean case. The analytic continuation to Lorenzian AdS_3 may be performed by Wick rotating one of the three spacelike coordinates in (2.4). The wavefunctions (2.6) present the semiclassical expressions for operators in the H_3^+ WZNW model. The OPEs of the SL(2,R) currents with each other encode the Kac-Moody algebra with central charge[2] k.

$$\begin{aligned} J^3(z)J^\pm(w) &\sim \frac{\pm J^\pm(w)}{z-w}, \\ J^3(z)J^3(w) &\sim -\frac{\frac{k}{2}}{(z-w)^2}, \\ J^-(z)J^+(w) &\sim \frac{k}{(z-w)^2}+\frac{2J^3(w)}{z-w}. \end{aligned} \tag{2.7}$$

These OPEs imply that the current modes defined by an expansion

$$J^a(z)=\sum_{n=-\infty}^{\infty}\frac{J^a_n}{z^{n+1}} \tag{2.8}$$

satisfy the following commutation relations

$$\begin{aligned} \left[J^3_n,J^3_m\right] &= -\frac{k}{2}n\delta_{n+m,0}, \\ \left[J^3_n,J^\pm_m\right] &= \pm J^\pm_{n+m}, \\ \left[J^+_n,J^-_m\right] &= -2J^3_{n+m}+kn\delta_{n+m,0}. \end{aligned} \tag{2.9}$$

The functions (2.6) are promoted to CFT operators $\Phi_h(y,\bar{y};w,\bar{w})$ which are primaries of the current algebra:

$$\begin{aligned} J^3(z)\Phi_h(y,\bar{y};w,\bar{w}) &\sim -\frac{(y\partial_y+h)\Phi_h(y,\bar{y};w,\bar{w})}{z-w}, \\ J^+(z)\Phi_h(y,\bar{y};w,\bar{w}) &\sim -\frac{(y^2\partial_y+2hy)\,\Phi_h(y,\bar{y};w,\bar{w})}{z-w}, \\ J^-(z)\Phi_h(y,\bar{y};w,\bar{w}) &\sim -\frac{\partial_y\Phi_h(y,\bar{y};w,\bar{w})}{z-w}. \end{aligned} \tag{2.10}$$

where $w,\bar{w}$ stand for the coordinates on the worldsheet. The stress-energy tensor follows from Sugawara construction

$$T=\frac{1}{k-2}[-(J^3)^2+J^+J^-]. \tag{2.11}$$

[2]The SL(2,R) WZNW model with central charge k describes bosonic string theory in AdS_3 of radius $\sqrt{k}$. In our geometric description of D-branes the coordinates are rescaled so that the resulting space is AdS_3 of unit radius.

The OPEs above imply that the global part of Kac-Moody symmetry can be interpreted as global conformal symmetry in spacetime. In fact, the correspondence extends to the full infinite-dimensional $\widehat{SL}(2,R)_k$ symmetry [18]. The operator $\Phi_h(y,\bar{y};w,\bar{w})$ is a primary under both the spacetime and the worldsheet conformal transformations with scaling dimensions h and

$$\Delta_h = -uh(h-1) \tag{2.12}$$

respectively. In the equation above we defined u as

$$u = \frac{1}{k-2}. \tag{2.13}$$

Similar formulae hold in the antiholomorphic sector.

An important ingredient of the H_3^+ WZNW model is an operator $\Phi_{-\frac{1}{2}}(y,\bar{y})$ which has a simple semiclassical form

$$\Phi_{-\frac{1}{2}}(y,\bar{y}) = \frac{2}{\pi}\left(|\gamma - y|^2 e^{\phi} + e^{-\phi}\right). \tag{2.14}$$

It is usually assumed that $\partial_y^2 \Phi_{-\frac{1}{2}}(y,\bar{y}) = 0$ holds as an operator equation. This implies that the operator $\Phi_{-\frac{1}{2}}$ is degenerate

$$[\Phi_{-\frac{1}{2}}]\,[\Phi_h] \sim [\Phi_{h+\frac{1}{2}}] + [\Phi_{h-\frac{1}{2}}]. \tag{2.15}$$

That is, the OPE of $\Phi_{-\frac{1}{2}}$ with a generic primary Φ_h contains only the current algebra blocks of $\Phi_{h+\frac{1}{2}}$ and $\Phi_{h-\frac{1}{2}}$.

Let us now briefly review the results of Ref. [2] which studies D-branes in AdS_3 background from the target space point of view. Depending on the gluing conditions for the holomorphic and antiholomorphic currents at the boundary of the worldsheet, possible D-branes in AdS_3 wrap conjugacy classes, twisted by a group automorphism that descends from an algebra automorphism used in gluing the currents. Recall that the conjugacy class twisted by the automorphism $w(h)$ is defined as

$$\mathcal{W}_g^w = \{w(h)gh^{-1}, \forall h \in SL(2,R)\,\}. \tag{2.16}$$

In the case of the inner automorphism, $w(h) = g_0^{-1}hg_0$, $g_0 \in SL(2,R)$, the set $\mathcal{W}_g^w$ is a left translation of the regular (untwisted) conjugacy class, so it is sufficient to consider the latter. In the parameterization (2.1) the regular conjugacy class is characterized by

$$\mathrm{tr}\,\mathrm{g} = 2\mathrm{X}^0 = 2\tilde{\mathrm{C}}. \tag{2.17}$$

It is a two-dimensional surface in the space (2.2) described by the following equation

$$(X^3)^2 - (X^1)^2 - (X^2)^2 = 1 - \tilde{C}^2. \tag{2.18}$$

Depending on the value of $\tilde{C}$, this surface can be a hyperbolic H_2^+ plane ($\tilde{C} < 1$), a lightcone ($\tilde{C} = 1$) or a de-Sitter dS_2 plane ($\tilde{C} > 1$). A coordinate system that will be useful for the description of dS_2 D-branes is

$$X^0 = \cosh\tilde{\psi}, \quad X^3 = \sinh\tilde{\psi}\sinh\tilde{t}, \quad X^1 + iX^2 = \sinh\tilde{\psi}\cosh\tilde{t}\, e^{i\phi}. \tag{2.19}$$

(this coordinate system covers part of Lorenzian AdS_3). The metric is given by

$$ds^2 = d\tilde{\psi}^2 + \sinh^2\tilde{\psi}(-d\tilde{t}^2 + \cosh^2\tilde{t}d\phi^2). \tag{2.20}$$

After substitution of the solutions of equations of motion, the DBI action of the D-brane located at constant

$$X^0 = \tilde{C} = \cosh\tilde{\psi}_0 \tag{2.21}$$

is given by [2]

$$I_{dS_2} = i\sinh\tilde{\psi}_0\, T_D \int d\tilde{t}d\phi\, \cosh\tilde{t} \tag{2.22}$$

where T_D is the (fixed) D-brane tension. The action is imaginary because there is a supercritical electric field living on the dS_2 brane [2]. H_2 D-branes with $\tilde{C} < 1$ are best described in the coordinate system

$$X^0 = \cos\tilde{\tau}, \qquad X^3 = \sin\tilde{\tau}\cosh\chi, \qquad X^1 + iX^2 = \sin\tilde{\tau}\sinh\chi e^{i\phi} \tag{2.23}$$

where the metric is

$$ds^2 = -d\tilde{\tau}^2 + \sin^2\tilde{\tau}(d\chi^2 + \sinh^2\chi d\phi^2). \tag{2.24}$$

In this coordinate system H_2 branes appear as surfaces of constant

$$X^0 = \tilde{C} = \cos\tilde{\tau}_0. \tag{2.25}$$

The DBI action of these branes is imaginary

$$I_{H_2} = iT_D\sin\tilde{\tau}_0 \int d\chi d\phi\, \sinh\chi. \tag{2.26}$$

Consider the fate of D-branes defined by (2.17), under the Wick rotation of $\tilde{X}^3$ (of course, one may equivalently Wick rotate X^1 or X^2). X^0 is now constrained as in (2.4), and therefore $\tilde{C} < 1$ is ruled out. dS_2 surfaces thus turn into two-spheres described by

$$(\tilde{X}^3)^2 + (X^1)^2 + (X^2)^2 = \tilde{C}^2 - 1, \tag{2.27}$$

while the lightcone becomes a point (degenerate two-sphere). The Euclidean action of such D-branes is given by (2.22). No extra factors of i appear, as dS_2 is a timelike surface in AdS_3. After the analytic continuation to H_3^+, the coordinate $\tilde{t}$ takes a finite range $\tilde{t} \in [0, 2\pi]$, and therefore the DBI action, which contains the volume of the D-brane, becomes finite.

Let us now look at conjugacy classes twisted by the outer isomorphism

$$w(h) = w_0^{-1} h w_0, \qquad \omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.28}$$

These are AdS_2 surfaces defined by

$$\begin{aligned} \mathrm{tr}\, \mathrm{g} = 2\mathrm{X}^2 &= 2C, && (2.29)\\ (X^0)^2 - (X^1)^2 + (X^3)^2 &= 1 + C^2. && (2.30) \end{aligned}$$

The convenient coordinate system

$$X^1 = \cosh\psi \sinh w, \quad X^2 = \sinh\psi, \quad X^0 + iX^3 = \cosh\psi \cosh w e^{it} \tag{2.31}$$

has the metric

$$ds^2 = d\psi^2 + \cosh^2\psi(-\cosh^2 w dt^2 + dw^2). \tag{2.32}$$

Note that the coordinate t has an infinite range in AdS_3. The DBI action of the AdS_2 D-brane located at constant

$$X^2 = C = \sinh\psi_0 \tag{2.33}$$

is now real [2]

$$I_{AdS_2} = \cosh\psi_0 \, T_D \int dt dw \, \cosh w. \tag{2.34}$$

The Euclidean counterparts of AdS_2 surfaces are H_2^+ planes described by

$$(X^0)^2 - (X^1)^2 - (\tilde{X}^3)^2 = 1 + C^2. \tag{2.35}$$

The Euclidean DBI action is still given by (2.34). It contains an infinite volume of the hyperbolic plane.

3 CFT boundary states

In this section we will review the conformal bootstrap of [1] and show how the crossing symmetry of the two-point function $\langle\Phi_{-\frac{1}{2}}\Phi_h\rangle$ on the disk leads to the constraint on the one-point function of the primary Φ_h. By solving this equation we obtain all allowed boundary states. We first discuss D-branes that have a finite spectrum of open strings living on them. We show that the complementary constraint equation that arises from the crossing symmetry of the "dual" two-point function $\langle\Phi_{\frac{k+1}{2}}\Phi_h\rangle$ implies that such D-branes are labeled by integers and correspond to degenerate representations of SL(2,R). Technically this follows from a simple relation between the original and the dual two-point functions, which is a consequence of the spectral flow symmetry of the underlying

SL(2,R)/U(1) coset. Hence, we re-derive the result, which was obtained in Ref. [1] who used the two-point function $\langle \Phi_{\frac{2-k}{2}} \Phi_h \rangle$ to derive the complementary equation for the one-point function. We next discuss boundary states that have a continuous spectrum of open strings living on them, and therefore correspond to extended D-branes in AdS_3. This discussion will be important in the next section, where we turn to the geometric description of such D-branes.

We start by considering the holomorphic part of the four-point function of primary operators on the sphere. The projective Ward identities for the worldsheet stress-energy tensor and for the currents constrain the four-point function to have the following form

$$\langle \Phi_{h_0}(y_0,z_0)\Phi_{h_1}(y_1,z_1)\Phi_{h_2}(y_2,z_2)\Phi_{h_3}(y_3,z_3)\rangle = \prod y_{ij}^{\mu_{ij}}\, z_{ij}^{\nu_{ij}}\, H\left(\begin{array}{cc} h_0 & h_1 \\ h_2 & h_3 \end{array}, y, z\right), \tag{3.1}$$

where $y_{ij} \equiv y_i - y_j$, y is the projective invariant

$$y = \frac{(y_0-y_1)(y_2-y_3)}{(y_0-y_3)(y_2-y_1)}, \tag{3.2}$$

and the non-zero μ_{ij} are

$$\mu_{03} = -2h_0, \tag{3.3}$$
$$\mu_{31} = -h_1 - h_3 + h_0 + h_2, \tag{3.4}$$
$$\mu_{32} = -h_2 - h_3 + h_0 + h_1, \tag{3.5}$$
$$\mu_{21} = -h_0 - h_1 - h_2 + h_3. \tag{3.6}$$

The worldsheet variables z_{ij}, z, and ν_{ij} are defined similarly, with the substitution $h_i \to \Delta_{h_i}$. The function $H\left(\begin{array}{cc} h_0 & h_1 \\ h_2 & h_3 \end{array}, y, z\right)$ satisfies the Knizhnik-Zamolodchikov equation. As usual, it can be derived from (2.11) and the current Ward identities, using the OPEs of the SL(2,R) currents with the primaries of the Kac-Moody algebra. The fact that the currents generate conformal symmetry in spacetime means that the equation can be conveniently written in terms of differential operators [19]. In our case the equation takes the form

$$\begin{aligned} \Big[&- z(z-1)(k-2)\partial_z + y(y-1)(z-y)\partial_y^2 \\ &+ \Big[(\Delta+1)(-y^2+2zy-z) - 2h_0 y(y-1) - 2yh_1(z-1) - 2(y-1)h_2 z\Big]\,\partial_y \\ &+ [2h_0\Delta(z-y) - 2h_0h_1(z-1) - 2h_0h_2 z]\Big]\, H\left(\begin{array}{cc} h_0 & h_1 \\ h_2 & h_3 \end{array}, y, z\right) = 0, \end{aligned} \tag{3.7}$$

where we introduced $\Delta = h_0 + h_1 + h_2 - h_3$. To define the theory on the disk[1], one must specify the gluing conditions for the holomorphic and antiholomorphic currents at the boundary of the worldsheet. Most of the subsequent discussion will be restricted to the diagonal gluing, which implies the following form of the one-point function

$$\langle \Phi_h(y,\bar{y};z,\bar{z})\rangle = \frac{U(h)}{(y-\bar{y})^{2h}\,|z-\bar{z}|^{2\Delta_h}} = \frac{\tilde{U}(h)}{|y-\bar{y}|^{2h}|z-\bar{z}|^{2\Delta_h}}. \tag{3.8}$$

In the equation above we define $\tilde{U}(h)$ and $U(h)$, which are related as

$$\begin{aligned} \tilde{U}(h) &= i^{-2h}U(h), \qquad \operatorname{Im} y > 0; \\ \tilde{U}(h) &= i^{2h}U(h), \qquad \operatorname{Im} y < 0. \end{aligned} \tag{3.9}$$

Note that because of (3.9) it is impossible to have both $\tilde{U}(h)$ and $U(h)$ to be completely y-independent. We will see that while $U(h)$ is y-independent, $\tilde{U}(h)$ depends on the sign of $(y-\bar{y})$ via (3.9).

In the boundary CFT with the diagonal gluing, the two-point function on the disk, which contains $\Phi_{-\frac{1}{2}}$, can be written as

$$\langle \Phi_{-\frac{1}{2}}(y_1,z_1)\Phi_h(y_2,z_2)\rangle = \frac{|y_2-\bar{y}_2|^{-1-2h}}{|y_1-\bar{y}_2|^{-2}}\,\frac{|z_2-\bar{z}_2|^{-\frac{3u}{2}-2\Delta_h}}{|z_1-\bar{z}_2|^{-3u}} H\left(\begin{array}{cc} -1/2 & h \\ -1/2 & h \end{array}, y, z\right), \tag{3.10}$$

where y and z are the spacetime and worldsheet cross-ratios

$$y = \frac{|y_1-y_2|^2}{|y_1-\bar{y}_2|^2}; \qquad z = \frac{|z_1-z_2|^2}{|z_1-\bar{z}_2|^2}. \tag{3.11}$$

One can solve the Knizhnik-Zamolodchikov equation[2] and write the two-point function as

$$\begin{aligned} \langle \Phi_{-\frac{1}{2}}(y_1,z_1)\Phi_h(y_2,z_2)\rangle &= \frac{|y_2-\bar{y}_2|^{-1-2h}}{|y_1-\bar{y}_2|^{-2}}\,\frac{|z_2-\bar{z}_2|^{-\frac{3u}{2}-2\Delta_h}}{|z_1-\bar{z}_2|^{-3u}} \\ &\times \left[C_+\tilde{U}(h+\frac{1}{2})\left(yH_1^+(z)+H_0^+(z)\right) + C_-\tilde{U}(h-\frac{1}{2})\left(yH_1^-(z)+H_0^-(z)\right)\right], \end{aligned} \tag{3.12}$$

where C_+ and C_- are the structure constants, and conformal blocks are given by

$$H_1^+ = z^{u(1-h)}(1-z)^{\frac{3u}{2}}F(u, 1+2u(1-h), 1+u(1-2h), z), \tag{3.13}$$

$$H_0^+ = \frac{uz^{1+u(1-h)}(1-z)^{\frac{3u}{2}}}{(1+u(1-2h))}F(1+u, 1+2u(1-h), 2+u(1-2h), z), \tag{3.14}$$

$$H_1^- = (2h-1)^{-1}z^{uh}(1-z)^{\frac{3u}{2}}F(1+u, 2hu, 1+u(2h-1), z), \tag{3.15}$$

$$H_0^- = z^{uh}(1-z)^{\frac{3u}{2}}F(u, 2hu, u(2h-1), z). \tag{3.16}$$

[1]All our calculations are performed on the upper half plane, which can be conformally mapped to the disk.

[2]See Appendix A for the solutions of the Knizhnik-Zamolodchikov equations in some special cases.

Here and elsewhere $F(A,B,C,z)$ is a hypergeometric function ${}_2F_1(A,B,C,z)$. To fix the coefficients of conformal blocks one should notice that the $|y_1-y_2|$ and $|z_1-z_2|$ dependence of the H_1^+ and H_0^- corresponds to the contribution of the operators $\Phi_{h+\frac{1}{2}}$ and $\Phi_{h-\frac{1}{2}}$, respectively. The two-point function on the disk enjoys the crossing symmetry. In Appendix A we show that in addition to (3.12), it can also be written as

$$\langle\Phi_{-\frac{1}{2}}(y_1,z_1)\Phi_h(y_2,z_2)\rangle = \frac{|y_2-\bar{y}_2|^{-1-2h}}{|y_1-\bar{y}_2|^{-2}}\frac{|z_2-\bar{z}_2|^{-\frac{3u}{2}-2\Delta_h}}{|z_1-\bar{z}_2|^{-3u}} \times\left[B^+(1-z)^{\frac{3u}{2}}z^{u(1-h)}F(u,1+2u(1-h),1+2u,1-z)(1-y)+\cdots\right], \tag{3.17}$$

where we only displayed the conformal block that corresponds to the contribution of the identity operator and its descendants arising from the fusion of the operators Φ_h and $\Phi_{-\frac{1}{2}}$ to the boundary of the worldsheet.

3.1 D-branes labeled by a discrete parameter

Let us first discuss D-branes that have a finite spectrum of open strings living on them. It will be convenient to assume that the one-point function is normalized, i.e. is divided by the partition sum on the disk. The conformal block that appears in (3.17) has a simple behavior when two bulk operators are taken close to the boundary of the worldsheet and spacetime. Namely, its asymptotic behavior corresponds to the fusion of bulk primaries to the identity operators at the boundary. The normalization of the one-point function implies that the coefficient of the conformal block factorizes as [1, 20]

$$B^+ = \tilde{U}(h)\tilde{U}(-\frac{1}{2}). \tag{3.18}$$

From the equality of (3.12) and (3.17), and the transformation properties of hypergeometric functions[1] one can derive a constraint equation for the one-point function. To do this, it is sufficient to match the terms containing y in (3.12) and (3.17). With the help of (D.7) it may be shown that H_1^+ and H_1^- contain terms whose worldsheet dependence is precisely the same as that of the conformal block in (3.17). Equating the coefficients gives the following relation

$$\tilde{U}(-\frac{1}{2})\tilde{U}(h) = \frac{\Gamma(-2u)}{\Gamma(-u)}\left[C_-\tilde{U}(h-\frac{1}{2})\frac{\Gamma(u(2h-1))}{\Gamma(2u(h-1))} - C_+\tilde{U}(h+\frac{1}{2})\frac{\Gamma(1+u(1-2h))}{\Gamma(1-2hu)}\right]. \tag{3.19}$$

The expressions for the structure constants in the convenient normalization were found in [21] (see also [22]) by the free field techniques.

$$C_+ = \frac{2}{\pi}\mathcal{R}\left(-\frac{1}{2}\right), \tag{3.20}$$

[1] See Appendix D for some properties of hypergeometric functions.

$$C_- \;=\; \frac{2}{\pi}\mathcal{R}\left(-\frac{1}{2}\right)\frac{\Gamma(2u(h-1))\Gamma(1+u(1-2h))}{\Gamma(1+2u(1-h))\Gamma(u(2h-1))}.$$

Here $\mathcal{R}(h)$ is the reflection amplitude

$$\mathcal{R}(h) = \frac{\Gamma(1+u(1-2h))}{\Gamma(1-u(1-2h))}, \tag{3.21}$$

which appears as the quantum mechanical correction to the reflection symmetry [22]

$$\Phi_h(y,\bar{y};z,\bar{z}) = \mathcal{R}(h)\frac{2h-1}{\pi}\int d^2y'\,|y-y'|^{-4h}\Phi_{1-h}(y',\bar{y}';z,\bar{z}). \tag{3.22}$$

At this point it is worth noting that eq. (3.19) is invariant under

$$\tilde{U}(h) \to i^{\pm 4h}\tilde{U}(h). \tag{3.23}$$

This is simply a reflection of the freedom that still exists in the description. Namely, one can choose either $U(h)$ or $\tilde{U}(h)$ (see (3.8) for the definition) to be independent of the sign of $(y-\bar{y})$. [Note that one cannot choose both of them to be completely y-independent because of (3.9)]. To fix this freedom we may use the reflection symmetry [1]. By taking the expectation value on the upper half plane of both sides of (3.22) it is not hard to see that it is $U(h)$ that must be y-independent. The condition of the reflection symmetry (3.22) takes the form

$$\frac{U(h)}{(y-\bar{y})^{2h}} = \mathcal{R}(h)\frac{2h-1}{\pi}\int d^2y'\,|y-y'|^{-4h}\,\frac{U(1-h)}{(y'-\bar{y}')^{2-2h}}. \tag{3.24}$$

The integral in the RHS of the equation above was computed in [1]; the result is

$$U(h) = -\mathcal{R}(h)U(1-h). \tag{3.25}$$

This equation must be taken with some caution, as it contains some regularized divergences. We will return to this subject later in the paper. It will be convenient to write $U(h)$ as

$$U(h) = f(h)\Gamma(1+u(1-2h)), \tag{3.26}$$

The eq. (3.25) translates into

$$f(h) = -f(1-h). \tag{3.27}$$

The constraint equation (3.19) for $U(h)$ takes the form

$$\pi\Gamma(1-u)f(-\frac{1}{2})f(h) = f(h-\frac{1}{2}) + f(h+\frac{1}{2}). \tag{3.28}$$

To derive this, we substituted (3.20), (3.21) and (3.26) into (3.19). The solution of (3.28) which respects (3.27) is

$$f(h) = -\frac{1}{\pi\Gamma(1-u)}\frac{\sin[u\pi(2h'-1)(2h-1)]}{\sin[u\pi(2h'-1)]}. \tag{3.29}$$

We will see that this solution is also consistent with the spectral flow symmetry of the theory. At this point h' is an arbitrary complex number. It was shown in [1] that h' takes discrete values

$$2h'-1 \in Z. \tag{3.30}$$

Below we are going to re-derive this result using the action of the spectral flow symmetry on the two-point function. We will also provide verification by a direct computation. The reader who is not interested in the details of this computation may skip to the beginning of the subsection 3.2.

The observation that will be important in the following is that $\Phi_{\frac{k+1}{2}}$ is a degenerate operator which has the following fusion rules

$$[\Phi_{\frac{k+1}{2}}][\Phi_h] \sim [\Phi_{\frac{k}{2}-h-\frac{1}{2}}] + [\Phi_{\frac{k}{2}-h+\frac{1}{2}}]. \tag{3.31}$$

One way to see that this is true is to notice that the Knizhnik-Zamolodchikov equation (3.7) takes a simple form with two conformal blocks if $h_3 = \frac{k+1}{2}$. This was noticed long ago in the SU(2) case [19], and we explain how this comes about in Appendix A. Another proof, which we will give below, utilizes the existence of the spectral flow symmetry. In order to show that $\Phi_{\frac{k+1}{2}}$ is degenerate and to find the fusion rules of $\Phi_{\frac{k+1}{2}}$ with other operators in the theory consider the state

$$|\theta\rangle = (J^+_{-1})^2|\frac{k+1}{2}\rangle^{hw}, \tag{3.32}$$

where $|\frac{k+1}{2}\rangle^{hw}$ defined as

$$|\frac{k+1}{2}\rangle^{hw} \equiv \Phi_{\frac{k+1}{2}}(0)|0\rangle \tag{3.33}$$

is a highest weight state, i.e.

$$J^a_n|\frac{k+1}{2}\rangle^{hw} = 0 \quad n \geq 1; \tag{3.34}$$

$$J^+_0|\frac{k+1}{2}\rangle^{hw} = 0. \tag{3.35}$$

We will also need a definition of the lowest weight state below: the state $|a\rangle^{lw}$ is a lowest weight state if the following conditions are met

$$J^a_n|a\rangle^{lw} = 0 \quad n \geq 1; \tag{3.36}$$

$$J^-_0|a\rangle^{lw} = 0. \tag{3.37}$$

We claim that $|\theta\rangle$ is a null state. This statement is actually related to the fact that

$$\partial_y^2 \Phi_{-1/2}(y) \simeq 0, \tag{3.38}$$

which means that the operator in the left-hand side of this equation can be set to zero in all correlation functions. Indeed (3.38) can be rewritten as

$$|\tilde{\theta}\rangle^{hw} = (J_0^-)^2 | -1/2\rangle^{hw} \simeq 0 \quad \text{or} \tag{3.39}$$
$$|\tilde{\theta}\rangle^{lw} = (J_0^+)^2 | -1/2\rangle^{lw} \simeq 0,$$

where

$$| -1/2\rangle^{lw} \equiv \lim_{y\to\infty} y^{-1} \Phi_{-1/2}(y)|0\rangle \tag{3.40}$$

and $|\tilde{\theta}\rangle^{lw}$, $|\tilde{\theta}\rangle^{hw}$ are the lowest and the highest weight states respectively. As it was shown in [23], the spectral flow by minus one unit maps the lowest weight state $|h\rangle^{lw}$ into the highest weight state $|\frac{k}{2} - h\rangle^{hw}$ and also J_0^+ into J_{-1}^+. The spectral flow will map $|\tilde{\theta}\rangle^{lw}$ into the highest weight state which is actually equal to $|\theta\rangle$. We arrive to the conclusion that $|\theta\rangle$, while being a descendant of the current algebra, is also a primary. In a unitary theory this statement would mean that the operator corresponding to $|\theta\rangle$ is null and can be set to zero in all correlation functions. One should be aware of the fact that in the context of the H_3^+ WZNW model this is just an assumption, which seems to be a part of the definition of the theory.

One can also show directly that $|\theta\rangle$ is a primary, i.e.

$$\begin{aligned} &J_n^a|\theta\rangle = 0; \quad \text{for} \quad n > 0 \\ &J_0^+|\theta\rangle = 0. \end{aligned} \tag{3.41}$$

It is easy to see using (2.9) that the only non-trivial checks that should be performed are the ones involving J_n^- and J_n^3 with $n = 1, 2$. Indeed let $n > 2$ and $f^{ab}{}_c$ be the structure constants of $sl(2, R)$ Lie algebra, then

$$J_n^a (J_{-1}^+)^2 |\frac{k+1}{2}\rangle^{hw} = [J_n^a, (J_{-1}^+)^2] |\frac{k+1}{2}\rangle^{hw} = \sum_b f^{a+}{}_b [J_{n-1}^b, J_{-1}^+]|\frac{k+1}{2}\rangle^{hw} = \tag{3.42}$$
$$\sum_{b,c} f^{a+}{}_b f^{b+}{}_c J_{n-2}^c |\frac{k+1}{2}\rangle^{hw} = 0.$$

Using (2.9) we can verify that (3.41) holds. Let us show that $J_1^-|\theta\rangle = 0$

$$J_1^-(J_{-1}^+)^2|\frac{k}{2} + \frac{1}{2}\rangle^{hw} = ((2J_0^3 + k)J_{-1}^+ + J_{-1}^+(2J_0^3 + k))|\frac{k}{2} + \frac{1}{2}\rangle^{hw} = 0. \tag{3.43}$$

The rest of the checks can be performed in a similar manner. So we conclude that under our assumption the operator corresponding to $|\theta\rangle$ is null and can be set to zero in all correlators. Now this result can be used to derive the OPEs of the operator $\Phi_{\frac{k+1}{2}}$ with other operators in the theory. Suppressing the worldsheet dependence and the antiholomorphic part, we have

$$\Phi_h(y)\Phi_{\frac{k+1}{2}}(0) = \sum_{h'} C^{h'}_{\frac{k+1}{2}\,h} y^{-\frac{k+1}{2}-h+h'}\Phi_{h'}(0) + \dots . \tag{3.44}$$

Applying $(J^+_{-1})^2$ to the (3.44) and using the OPE (2.10) we obtain the following constraint on the structure constants

$$\left[y^2\partial_y + 2hy\right]\left[y^2\partial_y + 2hy\right] C^{h'}_{\frac{k+1}{2}\,h} y^{-\frac{k+1}{2}-h+h'} = 0, \tag{3.45}$$

which can be written as

$$\left(-\frac{k+1}{2} + h + h'\right)\left(-\frac{k}{2} + \frac{1}{2} + h + h'\right) C^{h'}_{\frac{k+1}{2}\,h} = 0. \tag{3.46}$$

We see that (3.46) implies (3.31). Below we will show that this form of fusion rules is actually fixed by spectral flow and follows from the fusion rules (2.15) of the operator $\Phi_{-\frac{1}{2}}$.

Consider the three-point function

$$\langle \Phi_{h_1}(y_1;z_1)\Phi_{h_2}(y_2;z_2)\Phi_{h_3}(y_3;z_3)\rangle = D(h_1,h_2,h_3)\,\eta(h_1,h_2,h_3;y_1,y_2,y_3;z_1,z_2,z_3). \tag{3.47}$$

In this expression $\eta(h_1,h_2,h_3;y_1,y_2,y_3;z_1,z_2,z_3)$ contains the worldsheet and spacetime dependence, determined by the current and conformal Ward identities (see [22] for more details). The explicit form of $D(h_1,h_2,h_3)$ is [22]

$$D(h_1,h_2,h_3) = \frac{k-2}{2\pi^3}\frac{G(-h_1-h_2-h_3+1)G(-h_1-h_2+h_3)G(-h_1-h_3+h_2)G(-h_2-h_3+h_1)}{G(-1)G(1-2h_1)G(1-2h_2)G(1-2h_3)}, \tag{3.48}$$

where G is some special function which satisfies

$$\begin{aligned} G(x) &= G(-x-k+1), \\ G(x-1) &= \frac{\Gamma(1+\frac{x}{k-2})}{\Gamma(-\frac{x}{k-2})}G(x). \end{aligned} \tag{3.49}$$

In eq. (3.48) the normalization of [21] is implied (the original formulae of [22] contain certain prefactors that are set to unity in this normalization). Using (3.49) one can show that

$$\begin{aligned} \frac{D(\hat h_1,\hat h_2,\hat h_3)}{D(h_1,h_2,h_3)} &= \frac{\Gamma(1+u[1-2\hat h_1])\Gamma(1+u[1-2\hat h_2])}{\Gamma(1+u[1-2h_1])\Gamma(1+u[1-2h_2])}, \\ \hat h &\equiv \frac{k}{2} - h. \end{aligned} \tag{3.50}$$

In Appendix B we show that this equation, which may also be written as

$$D(h_1,h_2,h_3)=\sqrt{\frac{\langle\Phi_{h_1}\Phi_{h_1}\rangle\langle\Phi_{h_3}\Phi_{h_3}\rangle}{\langle\Phi_{\frac{k}{2}-h_1}\Phi_{\frac{k}{2}-h_1}\rangle\langle\Phi_{\frac{k}{2}-h_3}\Phi_{\frac{k}{2}-h_3}\rangle}}D(\frac{k}{2}-h_1,h_2,\frac{k}{2}-h_3), \tag{3.51}$$

follows from the spectral flow symmetry. (In the equation above $\langle\Phi_h\Phi_h\rangle$ is a two point function of the operator Φ_h stripped of the spacetime and worldsheet dependence). Note that $\langle\Phi_h\Phi_h\rangle$ is divergent due to the contribution of zero modes but the divergences will cancel out in (3.51). One can convince oneself that the structure constant that appears in the OPE of the degenerate operator $\Phi_{\frac{k+1}{2}}$ is related to $D(\frac{k+1}{2},h,h')$ in a simple way

$$C^{h'}_{\frac{k+1}{2}\,h}=\frac{D(\frac{k+1}{2},h,h')}{\langle\Phi_{h'}\Phi_{h'}\rangle}. \tag{3.52}$$

It is interesting to note that because of the divergence in $\langle\Phi_{h'}\Phi_{h'}\rangle$, the structure constant $C^{\frac{k}{2}-h'}_{\frac{k+1}{2}\,h}$ can only be nonzero when $D(\frac{k+1}{2},h,h')$ is divergent. Now using (3.51) we can obtain the following relation

$$C^{\frac{k}{2}-h'}_{\frac{k+1}{2}\,h}=\sqrt{\frac{\langle\Phi_{\frac{k+1}{2}}\Phi_{\frac{k+1}{2}}\rangle\langle\Phi_{h'}\Phi_{h'}\rangle}{\langle\Phi_{-\frac{1}{2}}\Phi_{-\frac{1}{2}}\rangle\langle\Phi_{\frac{k}{2}-h'}\Phi_{\frac{k}{2}-h'}\rangle}}C^{h'}_{-\frac{1}{2}\,h}. \tag{3.53}$$

Since the expression under the square root is non-zero and finite, $C^{\frac{k}{2}-h'}_{\frac{k+1}{2}\,h}$ and $C^{h'}_{-\frac{1}{2}\,h}$ are non zero at the same values of h and h'. Hence the fusion rules (3.31) indeed follow from (2.15).

Let us now consider the two-point function on the disk containing this degenerate operator. This two-point function reads

$$\begin{aligned}\langle\Phi_h(y_1,z_1)\Phi_{\frac{k+1}{2}}(y_2,z_2)\rangle \;=\; & |y_2-\bar{y}_2|^{-k-1+2h}|y_1-\bar{y}_2|^{-4h} \\ & |z_2-\bar{z}_2|^{2\Delta_h-2\Delta_{\frac{k+1}{2}}}|z_1-\bar{z}_2|^{-4\Delta_h}H\left(\begin{array}{cc} h & (k+1)/2 \\ h & (k+1)/2\end{array},y,z\right).\end{aligned} \tag{3.54}$$

The conformal block that appears in the equation above is given by

$$H\left(\begin{array}{cc} h & (k+1)/2 \\ h & (k+1)/2\end{array},y,z\right)=(y-z)^{-2h}H_0'(z)+(y-z)^{-2h-1}H_1'(z), \tag{3.55}$$

with H_0' and H_1' related to H_0 and H_1 as

$$\begin{aligned}H_0'(z)&=z^h(1-z)^{-2\Delta_h+2\Delta_{-1/2}}H_0(z),\\ H_1'(z)&=z^{h+1}(1-z)^{-2\Delta_h+2\Delta_{-1/2}}(H_0(z)+H_1(z)),\end{aligned} \tag{3.56}$$

where we omit possible normalization factors that are independent of the worldsheet coordinates. We explain how the equations above follow from the spectral flow symmetry in Appendix B, and verify them directly by solving the corresponding Knizhnik-Zamolodchikov equation in Appendix A. The two-point function (3.54) then takes the form that is similar to (3.12)

$$\begin{aligned}\langle\Phi_h(y_1,z_1)\Phi_{\frac{k+1}{2}}(y_2,z_2)\rangle &= |y_2-\bar y_2|^{-k-1+2h}|y_1-\bar y_2|^{-4h}|z_2-\bar z_2|^{2\Delta_h-2\Delta_{\frac{k+1}{2}}}|z_1-\bar z_2|^{-4\Delta_h} \\ &\left[\tilde C'_+\tilde U(\frac{k}{2}-[h+\frac{1}{2}])\left[(y-z)^{-2h}H_0'^{+}(z)+(y-z)^{-2h-1}H_1'^{+}(z)\right]+\right. \\ &\left.\tilde C'_-\tilde U(\frac{k}{2}-[h-\frac{1}{2}])\left[(y-z)^{-2h}H_0'^{-}(z)+(y-z)^{-2h-1}H_1'^{-}(z)\right]\right],\end{aligned} \tag{3.57}$$

where $\tilde C'_+$ and $\tilde C'_-$ are the structure constants, and the conformal blocks are given by

$$H_0'^{+} = z^{h+uh}(1-z)^{2uh(h-1)}F(u,2hu,u(2h-1),z), \tag{3.58}$$

$$H_1'^{+} = \frac{2h}{2h-1}z^{1+h+uh}(1-z)^{1+2uh(h-1)},F(1+u,1+2hu,1+u(2h-1),z), \tag{3.59}$$

$$H_0'^{-} = \frac{u\,z^{1+h+u(1-h)}(1-z)^{2uh(h-1)}}{1+u(1-2h)}F(1+u,1+2u(1-h),2+u(1-2h),z), \tag{3.60}$$

$$H_1'^{-} = z^{1+h+u(1-h)}(1-z)^{1+2uh(h-1)}F(1+u,1+2u(1-h),1+u(1-2h),z). \tag{3.61}$$

To obtain the dual equation for the one-point function we can again use the crossing symmetry. The two-point function (3.54) can also be written as

$$\begin{aligned}\langle\Phi_h(y_1,z_1)\Phi_{\frac{k+1}{2}}(y_2,z_2)\rangle &= |y_2-\bar y_2|^{-k-1+2h}|y_1-\bar y_2|^{-4h} \\ &\times|z_2-\bar z_2|^{2\Delta_h-2\Delta_{\frac{k+1}{2}}}|z_1-\bar z_2|^{-4\Delta_h}H\begin{pmatrix} h & h \\ (k+1)/2 & (k+1)/2\end{pmatrix}_{,y,z}.\end{aligned} \tag{3.62}$$

The analog of (3.17) in this case is

$$\begin{aligned}\langle\Phi_h(y_1,z_1)\Phi_{\frac{k+1}{2}}(y_2,z_2)\rangle&=|y_2-\bar y_2|^{-k-1+2h}|y_1-\bar y_2|^{-4h} \\ &\times\left[B'^{+}([1-y]-[1-z])^{-2h}(1-z)^{2uh(h-1)}z^{h+uh}F(u,2uh,1+2u,1-z)+\cdots\right].\end{aligned} \tag{3.63}$$

Now we can use the same technology that was employed in the derivation of (3.19). The only difference is that we now need to match terms containing $(y-z)$ in the expressions (3.57) and (3.63). It is also convenient to make a shift $h\to k/2-h$ in the resulting equation, which then takes the form

$$\begin{aligned}&(-)^{k-2h+1}\tilde U(\frac{k+1}{2})\tilde U(\frac{k}{2}-h) = \\ &\frac{\Gamma(-2u)}{\Gamma(-u)}\left[C'_-\tilde U(h-\frac{1}{2})\frac{\Gamma(u(2h-1))}{\Gamma(2u(h-1))}-C'_+\tilde U(h+\frac{1}{2})\frac{\Gamma(1+u(1-2h))}{\Gamma(1-2hu)}\right],\end{aligned} \tag{3.64}$$

where C'_+ and C'_- are the structure constants that appear in the OPE of $\Phi_{\frac{k+1}{2}}$ with $\Phi_{\frac{k}{2}-h}$

$$[\Phi_{\frac{k+1}{2}}]\,[\Phi_{\frac{k}{2}-h}] \sim C'_-[\Phi_{h-\frac{1}{2}}] + C'_+[\Phi_{h+\frac{1}{2}}]. \tag{3.65}$$

To find the relation between $C'_\pm$ and $C_\pm$ we may use (3.48), which gives

$$\begin{aligned} \frac{C'_\pm}{C_\pm} &= \frac{\Gamma(1+u[1-2\hat{h}_1])\Gamma(1+u[1-2\hat{h}_2])}{\Gamma(1+u[1-2h_1])\Gamma(1+u[1-2h_2])}, \\ h_1 &= \frac{k+1}{2}, \qquad h_2 = \frac{k}{2} - h, \\ \hat{h}_i &\equiv \frac{k}{2} - h_i. \end{aligned} \tag{3.66}$$

Substituting

$$\tilde{U}(h) = i^{\pm 2h}\Gamma(1+u[1-2h])\, f(h), \tag{3.67}$$

which is equivalent to (3.26), into (3.64) and (3.19), and using (3.66) one can infer that

$$f(\frac{k}{2} - h) = \pm f(h). \tag{3.68}$$

Applied to (3.29), this leads to the result (3.30).

3.2 Extended D-branes

In section 2 we encountered D-branes that wrap conjugacy classes in AdS_3 and therefore are extended in the target space. This implies that the spectrum of open strings living on such D-branes is continuous. It was proposed in [1] that the factorization property (3.18) is no longer valid for such D-branes and one should follow the lines of [24]. That is, one should consider *unnormalized* one-point function $U(h)$, rather than the normalized one, which played a central role in the description of D-branes in the previous subsection. Further, the coefficient B^+ in the boundary expansion of the two-point function (3.17) takes the form

$$B^+ = iA_0\tilde{U}(h), \tag{3.69}$$

where A_0 is the fusion coefficient of the operator $\Phi_{-\frac{1}{2}}$ to the identity operator on the boundary

$$\Phi_{-\frac{1}{2}}(y,\bar{y};w,\bar{w}) = A_0\frac{y-\bar{y}}{|w-\bar{w}|^{-\frac{3u}{2}}} + \cdots. \tag{3.70}$$

The identity (3.69) implies that the one-point function is again of the form (3.26) with $f(h)$ satisfying the equation

$$CA_0 f(h) = f(h-\frac{1}{2}) + f(h+\frac{1}{2}), \tag{3.71}$$

where C is some real number, whose precise value will not be needed in our discussion. The solution of this equation that respects the reflection symmetry (3.27) is

$$f(h) = A \sin\left[\Theta(2h-1)\right], \tag{3.72}$$

where A is some prefactor. The value of A may in principle be fixed via perturbative computation of the one-point function $U(h)$. The analogous computation in Liouville theory [24] implies that A is independent of the parameter that labels the D-brane. This observation was used in Ref. [1], who argued that this is also the case in the H_3^+ WZNW model. We will see that this conjecture is necessary for the consistency of the CFT and spacetime descriptions. Eq. (3.71) implies that Θ and A_0 are related as

$$CA_0 = 2\cos\Theta. \tag{3.73}$$

It is interesting to note that although (3.27), and hence (3.72) were derived for D-branes that preserve the diagonal SL(2,R) symmetry, they are also valid in case of other gluing conditions on the SL(2,R) currents at the boundary of the worldsheet. The gluing conditions that give rise to $\langle \Phi_h(y,\bar{y})\rangle = \frac{U(h)}{(1+y\bar{y})^{2h}}$ will be of particular importance. The analog of (3.24) in this case is

$$\frac{U(h)}{(1+y\bar{y})^{2h}} = \mathcal{R}(h)\frac{2h-1}{\pi}\int d^2y'\,|y-y'|^{-4h}\,\frac{U(1-h)}{(1+y'\bar{y}')^{2-2h}}. \tag{3.74}$$

We show that it leads to (3.27) in Appendix C.

We conclude this section by a remark about the parameter Θ which appears in the one-point function (3.72). Let us consider the annulus partition function of an open string stretched between an extended D-brane labeled by Θ and a fundamental D-instanton (the basic brane with $2h'-1=-1$). In the closed string channel this partition function can be written as

$$\begin{aligned}
Z_{(1,\Theta)} = \langle 1|e^{-2\pi T\mathcal{H}}|\Theta\rangle \quad &\sim \quad \int_{C^+} dh \int d^2y \frac{U(h)_1}{(1+y\bar{y})^{2h}}\,\frac{U(h)_\Theta}{(1+y\bar{y})^{2-2h}}\,\frac{q_c^{-u(h-\frac{1}{2})^2}}{\eta^3(q_c)}, \qquad (3.75)\\
q_c \quad &= \quad \exp(-2\pi T),\\
\eta(q_c) \quad &= \quad q_c^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q_c^n),\\
C^+ \quad &\equiv \quad \frac{1}{2}+iR.
\end{aligned}$$

Here T and $\mathcal{H}$ are the time and the Hamiltonian respectively, and the equality follows from inserting the complete set of closed string states into the matrix element. The subscripts in the one-point functions indicate the boundary states. We also assumed that both D-branes carry the same gluing conditions, which correspond to the one-point function of the form

$$\langle \Phi_h(y,\bar{y})\rangle = \frac{U(h)}{(1+y\bar{y})^{2h}}. \tag{3.76}$$

This behavior of the one-point function suggests that the corresponding D-brane does not introduce a boundary to the Euclidean spacetime. Further, with this choice of gluing conditions one does not encounter divergences in the derivation of eq. (3.27), which appear for other gluing conditions. All of this seems to imply that the D-instantons are described by the one-point function of the form (3.76) with $U(h)$ given by (3.29) and (3.26).

Let us continue to analyze (3.75). It is clear that no additional h-dependent factor comes from the d^2y integration. Hence, (3.75) reduces to

$$Z_{(1,\Theta)} \sim \int_0^\infty d\lambda\, \lambda \sinh[2\Theta\lambda] \exp(-2\pi T u\lambda^2)\, \eta^{-3}(q_c), \tag{3.77}$$

which is essentially what has been computed in [1], who argued that it admits an interpretation in the open string channel as a character of the principal continuous (principal discrete) representation of $\widehat{SL}(2,R)$ algebra if Θ is imaginary (real). In this case

$$\Theta = \frac{\pi(2h'-1)}{k-2} \tag{3.78}$$

is related to the spin $j' = h' - 1$ representation of SL(2,R) [1]. The situation is more complicated for boundary states which carry other gluing conditions, as an h-dependent term will modify the modular bootstrap (3.75). For example, an analog of (3.77) for the D-brane that is described by the one-point function of the form $\langle \Phi_h(y,\bar{y})\rangle = \frac{U(h)}{(y-\bar{y})^{2h}}$, will contain the additional h-dependent factor that is computed in Appendix C. The annulus partition function takes the form

$$Z_{(1,\Theta)} \sim \int_0^\infty d\lambda\, \left(\cosh[(2\Theta+\pi)\lambda] - \cosh[(2\Theta-\pi)\lambda]\right) \exp(-2\pi T u\lambda^2)\, \eta^{-3}(q_c). \tag{3.79}$$

This expression is not easily interpreted in the open string channel. We will comment on this result later in the paper.

4 Geometric interpretation

In this section we consider D-branes in H_3^+ that appear as surfaces of constant X^0 (two-spheres) and $\tilde{X}^3$ (hyperbolic planes)[1]. By Wick rotating one of the spacelike coordinates

[1] See section 2 for the discussion of the geometry of H_3^+, AdS_3, and D-branes wrapped on conjugacy classes.

in H_3^+, one may turn two-spheres into dS_2 surfaces. The fate of a D-brane at $\tilde{X}^3 = const$ depends on whether the coordinate which is Wick rotated lies within the worldvolume of the D-brane (X^1 or X^2) or not ($\tilde{X}^3$). In the first case the D-brane becomes the AdS_2 brane in AdS_3. In the second case it is necessary to have an imaginary value of $\tilde{X}^3$ for analytic continuation to make sense. Depending on the value of $|\tilde{X}^3|$, the resulting surface can be a dS_2 conjugacy class ($|\tilde{X}^3| > 1$), a light cone ($|\tilde{X}^3| = 1$), or an H_2 plane ($|\tilde{X}^3| < 1$). We do not consider D-branes located at $X^1 = const$ or at $X^2 = const$, since these two coordinates are on equal footing with $\tilde{X}^3$ in H_3^+.

What we will show is that all of D-branes mentioned above correspond to boundary states in CFT on the disk [1], which were reviewed in the previous section. We will see that the string partition sum on the disk reproduces the spacetime DBI action of corresponding D-branes. The technique that we will be using is based on the fact that the location of the brane in H_3^+ may be inferred from its boundary wavefunction[2]. In the computations below excited states of the string are neglected, so everything boils down to quantum mechanics. The quantum mechanical boundary wavefunction $\langle g|h'\rangle$ is a coordinate representation of the boundary state $|h'\rangle$. The coordinate states $|g\rangle$ are normalized to a delta-function which is defined with respect to the SL(2,R) invariant measure on H_3^+

$$\langle g|g'\rangle = \delta(g-g'); \qquad g, g' \in H_3^+. \tag{4.1}$$

A useful basis for normalizable functions in H_3^+ is provided by the wavefunctions defined in (2.6) [22, 25]

$$\langle h, y, \bar{y}|g\rangle = \tilde{\Phi}_h(y, \bar{y}|g). \tag{4.2}$$

The corresponding states satisfy the completeness relation:

$$\begin{aligned} &\int_{C^+} dh \int d^2y \, |h, y, \bar{y}\rangle\langle h, y, \bar{y}| = \mathbf{1}, \\ &C^+ \equiv \frac{1}{2} + iR. \end{aligned} \tag{4.3}$$

Using (4.3), the boundary wavefunction may be written as

$$\langle g|h'\rangle = \int_{C^+} dh \int d^2y \; \tilde{\Phi}_h(y, \bar{y}|g)\langle h, y, \bar{y}|h'\rangle. \tag{4.4}$$

It is natural to identify

$$\langle h, y, \bar{y}|h'\rangle = \langle \Phi_h(y, \bar{y})\rangle_{h'}. \tag{4.5}$$

[2]The similar computation in SU(2) gives D-branes that correspond to conjugacy classes which in this case are simply two-spheres [14]. Recently similar techniques were used for studying D-branes in AdS_3 [5].

where the expression in the right-hand side stands for the one-point function of $\Phi_h(y,\bar{y})$, stripped of the dependence of worldsheet coordinates. The $y,\bar{y}$-dependence is determined by the gluing conditions for the holomorphic and antiholomorphic currents.

4.1 D-Branes at $X^0 = const$

Let us consider the gluing that gives rise to the following one-point function

$$\langle \Phi_h(y,\bar{y})\rangle_{h'} = \frac{U(h)_{h'}}{(1+y\bar{y})^{2h}}, \tag{4.6}$$

where

$$U(h)_{h'} = A \sin\left[\frac{\pi(2h'-1)(2h-1)}{k-2}\right]\Gamma(1+\frac{1-2h}{k-2}). \tag{4.7}$$

We used (3.26), (3.72) and (3.78) in writing the formula (4.7) for the one-point function. To determine the location of the D-brane, which is described by (4.6)–(4.7), it is necessary to compute the overlap (4.4). Plugging in (4.7), (4.6) and (2.6) we can write (4.4) as

$$\langle g|h'\rangle = \int_{C^+} dh\, \frac{1-2h}{\pi}\int d^2y\, \frac{U(h)}{(1+y\bar{y})^{2h}} \frac{1}{\left(X^0{+}X^1{+}(X^0{-}X^1)y\bar{y}{+}2X^2y_1{+}2\tilde{X}^3y_2\right)^{2(1-h)}}. \tag{4.8}$$

Let us introduce $R=\sqrt{(X^2)^2{+}(\tilde{X}^3)^2}$. The integral over the y-plane can be rewritten as

$$\int d^2y\, \frac{1}{(1+y\bar{y})^{2h}} \frac{1}{\left(X^0{+}X^1{+}(X^0{-}X^1)y\bar{y}{+}2X^2y_1{+}2\tilde{X}^3y_2\right)^{2(1-h)}} = \tag{4.9}$$
$$(X^0{-}X^1)^{2h}\int d^2y\, \left((X^0{-}X^1)^2{+}y_1^2{+}y_2^2\right)^{-2h}\left(1{+}R^2{+}y_1^2{+}2Ry_1{+}y_2^2\right)^{2(h-1)},$$

where we defined $y = y_1 + iy_2$. Using the identity

$$\int_0^\infty dt\, t^\beta e^{-\alpha t} = \alpha^{-\beta-1}\Gamma(\beta+1), \tag{4.10}$$

we can rewrite (4.9) as

$$\frac{(X^0{-}X^1)^{2h}}{\Gamma(2h)\Gamma(2-2h)}\int d^2y \int_0^\infty dtds\, t^{2h-1}s^{1-2h} \tag{4.11}$$
$$\exp\left(-t[(X^0{-}X^1)^2{+}y_1^2{+}y_2^2] - s[1{+}R^2{+}2Ry_1{+}y_1^2{+}y_2^2]\right).$$

Completing the square and integrating over d^2y gives

$$\frac{(X^0-X^1)^{2h}\,\pi}{\Gamma(2h)\Gamma(2-2h)}\int_0^\infty dtds\, \frac{t^{2h-1}s^{1-2h}}{t+s}\exp\left(\frac{R^2s^2}{t+s} - t(X^0-X^1)^2 - s(1+R^2)\right). \tag{4.12}$$

Writing $t = \alpha s$ and integrating over s we obtain

$$\frac{(X^0 - X^1)^{2h}\pi}{\Gamma(2h)\Gamma(2-2h)} \int_0^\infty d\alpha \frac{\alpha^{2h-1}}{1 + \alpha(1+R^2) + (1+\alpha)\alpha(X^0 - X^1)^2}. \tag{4.13}$$

With the help of (2.4) this can be simplified to

$$\frac{\pi}{\Gamma(2h)\Gamma(2-2h)} \int_0^\infty d\alpha \frac{\alpha^{2h-1}}{1 + 2\alpha X^0 + \alpha^2}. \tag{4.14}$$

Note that this integral depends only on X^0. Recall that in H_3^+ $X^0 > 1$. Writing

$$X^0 = \cosh \tilde{\psi}, \tag{4.15}$$

and using the identity [26]

$$\int_0^\infty \frac{x^{\mu-1} dx}{(x+\beta)(x+\gamma)} = \frac{1}{\gamma - \beta}(\beta^{\mu-1} - \gamma^{\mu-1}) \frac{\pi}{\sin(\pi\mu)} \tag{4.16}$$
$$|\arg \beta|, |\arg \gamma| < \pi, \quad 0 < \mathrm{Re}\mu < 2,$$

the integral (4.9) becomes

$$\frac{\pi}{2h-1} \frac{\sinh[(2h-1)\tilde{\psi}]}{\sinh \tilde{\psi}}. \tag{4.17}$$

Substituting this into (4.8) we obtain the following expression for the boundary wave-function

$$\langle g | h' \rangle = -A \int_{C+} dh\, U(h) \frac{\sinh[(2h-1)\tilde{\psi}]}{\sinh \tilde{\psi}}. \tag{4.18}$$

We may now substitute (4.7) into this expression and perform the integration. Before we do this, however, let us note that although the integral formally runs over $h = \frac{1}{2} + i\lambda$, $\lambda \in R^+$, our analysis is only justified in the semiclassical limit, and therefore the terms of the order $h/(k-2)$ that appear in the argument of the Gamma function in (4.7) should be neglected. (This is similar to what happens in the SU(2) case [14].) For the D-brane labeled by

$$h' = \frac{1}{2} + i\lambda', \tag{4.19}$$

the boundary wavefunction becomes

$$\langle g | h' \rangle = -\frac{A}{\sinh \tilde{\psi}} \int_0^\infty d\lambda \, \sin[\frac{2\pi\lambda'}{k-2}\lambda] \sin[\tilde{\psi}\lambda]. \tag{4.20}$$

Up to an inessential numerical prefactor, this is simply a delta-function

$$\langle g | h' \rangle \sim A \frac{\delta(\tilde{\psi} - \frac{2\pi\lambda'}{k-2}) - \delta(\tilde{\psi} + \frac{2\pi\lambda'}{k-2})}{2\sinh \tilde{\psi}} = A\, \delta(X^0 - \cosh \tilde{\psi}_0), \tag{4.21}$$

where we defined

$$\tilde{\psi}_0 = \frac{2\pi\lambda'}{k-2}. \tag{4.22}$$

That is, the boundary state labeled by a parameter from the principal continuous series (4.19), gives rise to the D-brane which appears as the surface of constant

$$X^0 = \tilde{C} = \cosh\tilde{\psi}_0 \tag{4.23}$$

(equivalently, the two-sphere of radius $\sqrt{\tilde{C}^2 - 1}$) in H_3^+. The CFT partition function

$$U(h=0)_{h'} \sim i\sinh[\frac{2\pi\lambda'}{k-2}] = i\sinh\tilde{\psi}_0, \tag{4.24}$$

reproduces (2.22) up to a real normalization constant.

Let us now look at the fate of the D-brane described by (4.19) under the analytic continuation $\tilde{X}^3 \to iX^3$. The two-sphere $X^0 = \tilde{C}$ becomes the dS_2 surface in Lorenzian AdS_3. The important difference from the Euclidean case is that X^0 becomes an unrestricted coordinate. In the computation of the matrix element above, we used the positivity of $(X^0 - X^1)$, which is equivalent to the positivity of X^0. It is not hard to see that under the change of sign in X^0, the integrand in (4.8) picks up an h-dependent phase $(-)^{2h}$. As was explained in the previous section, conformal bootstrap is insensitive to such a phase. Therefore, in light of the above discussion, it seems natural to propose that the multiplication of the one-point function by $(-)^{2h}$ corresponds to the reflection $X^0 \to -X^0$ in the target space. In other words, the one-point functions that correspond to D-branes located at $X^0 = \tilde{C}$ and at $X^0 = -\tilde{C}$ differ by $(-)^{2h}$.

Before we proceed to other gluing conditions, let us comment on the fate of D-branes defined by (4.6) as above but with h' not belonging to C^+. For an illustration, let us take $h' = \frac{1}{2} + \mu'$ to be real. Then the boundary wavefunction (4.18) turns into

$$\langle g|h'\rangle = -\frac{A}{\sinh\tilde{\psi}}\int_0^\infty d\lambda\ \sinh[\frac{2\pi\mu'\lambda}{k-2}]\sin[\tilde{\psi}\lambda]. \tag{4.25}$$

This is a divergent integral with the integrand being an oscillating function whose amplitude increases exponentially as $\lambda \to \infty$. The only sensible answer for this integral is zero. One can see this directly by rewriting the integrand as a sum of exponents and using analytic continuation. This is also physically reasonable since, as we will see, D-branes with real h' appear as H_2 surfaces in Lorenzian AdS_3 and therefore should not appear in H_3^+.

It is interesting to observe that D-branes with h' real, prohibited in the Euclidean case, become allowed after the analytic continuation to Lorenzian AdS_3. In the parameterization

$$X^0 = \cos\tilde{\tau}, \tag{4.26}$$

the boundary wavefunction (4.18) becomes

$$\langle g|h'\rangle = -\frac{A}{\sin\tilde{\tau}}\int dh\ \sin[\frac{2\pi\mu'(2h-1)}{k-2}]\sin[(2h-1)\tilde{\tau}]. \tag{4.27}$$

It is clear that it is necessary to rotate the contour of integration to the real axis in order for this integral to make sense. This is quite reasonable, since in Lorenzian AdS_3 the normalizable wavefunctions include the highest and lowest weight states with real $h \geq 1/2$. The integral over h therefore corresponds to the trace over these states. Performing the integral gives

$$\langle g|h'\rangle \sim A\,\frac{\delta(\tilde{\tau}-\frac{2\pi\mu'}{k-2})}{\sin\tilde{\tau}}. \tag{4.28}$$

Hence we recover the H_2^+ brane located at $X^0 = \cos[\frac{2\pi\mu'}{k-2}] = \cos\tilde{\tau}_0$. The string partition sum

$$U(h=0)_{h'} \sim \sin\tilde{\tau}_0, \tag{4.29}$$

reproduces the corresponding DBI action (2.26).

4.2 D-Branes at $X^3 = const$

Let us now explore the gluing conditions that give rise to the following one-point function

$$\langle\Phi_h(y,\bar{y})\rangle_\Theta = \frac{U(h)_\Theta}{(y-\bar{y})^{2h}}, \tag{4.30}$$

where

$$U(h)_\Theta = A\sin[\Theta(2h-1)]\,\Gamma(1+\frac{1-2h}{k-2}). \tag{4.31}$$

In the equations above we introduced a complex parameter Θ, which labels the boundary state, in accord with the discussion at the end of the previous section. The boundary wavefunction now takes the form

$$\langle g|\Theta\rangle = \int_{C^+} dh\ \frac{1-2h}{\pi}\int d^2y\ \frac{U(h)}{(2iy_2)^{2h}}\frac{1}{\left(X^0+X^1+(X^0-X^1)y\bar{y}+2X^2y_1+2\tilde{X}^3y_2\right)^{2(1-h)}}. \tag{4.32}$$

After integration over y_1 the result depends only on $\tilde{X}^3$:

$$\langle g|\Theta\rangle = \frac{1-2h}{\pi}\frac{\Gamma(\frac{3}{2}-2h)\Gamma(\frac{1}{2})}{\Gamma(2-2h)}\int_{C^+} dh\ U(h)\int dy_2\ (2iy_2)^{-2h}\left[(y_2+\tilde{X}^3)^2+1\right]^{-\frac{3}{2}+2h}. \tag{4.33}$$

Using (4.10), completing the square and performing the integral over y_1 we obtain

$$\frac{\Gamma(\frac{3}{2}-2h)\Gamma(\frac{1}{2})}{\Gamma(2-2h)}\int dy_2\,(2iy_2)^{-2h}\left[(y_2+\tilde{X}^3)^2+1\right]^{-\frac{3}{2}+2h}= \tag{4.34}$$
$$\frac{\pi}{\Gamma(2h)\Gamma(\frac{3}{2}-2h)}\int_0^\infty dtds\,t^{2h-1}s^{-2h}\exp(-s+2i\tilde{X}^3t-t^2/s).$$

Writing $t=\alpha s$ and integrating over s we obtain

$$\frac{\pi}{\Gamma(2h)\Gamma(2-2h)}\int_0^\infty d\alpha\frac{\alpha^{2h-1}}{1-2i\alpha\tilde{X}^3+\alpha^2}. \tag{4.35}$$

which is very similar to (4.14). We performed some formal manipulations, so it would be nice to have an independent check of the formula above. Fortunately, there is a straightforward way of relating (4.35) to the computations we have done previously. Namely, we can set $\tilde{X}^3=iX^3$ to be imaginary. This converts (4.35) into (4.14) with the substitution $X^3\to X^2$. Therefore we indeed recover dS_2 and H_2 branes of the previous subsection. As in the case of dS_2 (H_2) branes at $X^0=const$, the identification (3.78) relates the complex parameter in the one-point function to the principal continuous (discrete) representation of SL(2,R).

For real $\tilde{X}^3$ let us introduce the parameterization

$$\tilde{X}^3=\sinh\psi. \tag{4.36}$$

The integral (4.35) becomes

$$\frac{\pi}{\Gamma(2h)\Gamma(2-2h)}\int_0^\infty d\alpha\frac{\alpha^{2h-1}}{(\alpha+e^{\psi-i\frac{\pi}{2}})(\alpha+e^{-(\psi-i\frac{\pi}{2})})}. \tag{4.37}$$

Using (4.16) this can be written as

$$\frac{\pi}{2h-1}\,\frac{\sinh[(2h-1)(\psi-i\frac{\pi}{2})]}{\cosh\psi}, \tag{4.38}$$

and therefore the boundary wavefunction becomes

$$\langle g|\Theta\rangle=-\frac{A}{\cosh\psi}\int_0^\infty d\lambda\;\sin[i\Theta\lambda]\sin[\lambda(\psi-i\frac{\pi}{2})]. \tag{4.39}$$

When Θ is of the form

$$\Theta=i\psi_0+\frac{\pi}{2}, \tag{4.40}$$

the integral in (4.39) produces the delta-function

$$\langle g|\Theta\rangle\sim A\,\frac{\delta(\psi-\psi_0)}{\cosh\psi_0}. \tag{4.41}$$

Hence we recover the H_2 brane located at

$$\tilde{X}^3 = C = \sinh\psi_0, \tag{4.42}$$

which becomes the AdS_2 brane after the Wick rotation of X^1 or X^2. Note that in this case the spacetime reflection $\tilde{X}^3 \to -\tilde{X}^3$ corresponds to the complex conjugation of Θ in the worldsheet description. The string partition sum on the disk

$$U(0)_\Theta \sim \sin[i\psi_0 + \frac{\pi}{2}] = \cosh\psi_0, \tag{4.43}$$

reproduces the DBI action (2.34) up to some *real* normalization factor.

To perform another consistency check of (4.40) let us recall the construction of [1]. According to [1], various boundary states in the H_3^+ WZNW model on the disk correspond to the value of boundary perturbation constant E which enters the boundary CFT action. The latter can be written in the free-field representation as

$$\mathcal{S}_\mathcal{B} = E\int dz \beta e^{-\phi(z)}, \tag{4.44}$$

where the integral runs over the real line, which is the boundary of the upper half plane. The free-field (Wakimoto) representation of the H_3^+ WZNW model contains the linear dilaton field ϕ, together with the (β, γ) system of conformal weight (1,0). The details of the Wakimoto representation will not be important to us, and we only note that the value of the (k-dependent) slope of the linear dilaton theory is consistent with the worldsheet and spacetime behavior of correlators demanded by Ward identities. The OPE (3.70) can be derived perturbatively, by expanding the exponential of the boundary action to the first order [1]. Using the free-field representation of the operator

$$\Phi_{-\frac{1}{2}}(y,\bar{y};w,\bar{w}) = \frac{2}{\pi}\mathcal{R}(-1/2)\gamma(w)e^{\phi(w)}y + \cdots \tag{4.45}$$

gives the following equation

$$\frac{A_0}{|w-\bar{w}|^{-\frac{3u}{2}}} = E\int dz\,\beta(z)e^{-\phi(z)}\;\frac{2}{\pi}\mathcal{R}(-1/2)\gamma(w)e^{\phi(w)}. \tag{4.46}$$

Here $\mathcal{R}(-1/2)$ is a real reflection coefficient necessary to account for the multiplicative renormalization of the operator $\Phi_{-\frac{1}{2}}$. By using the free field OPEs, it is not hard to see that the right hand side of (4.46) is a pure imaginary number. Therefore A_0 must be pure imaginary. Interestingly, the relation (3.73) implies that this can only be the case if Θ is of the form (4.40).

4.3 Volume divergences

The expressions (2.22) and (2.34) contain divergent integrals over the worldvolume of corresponding conjugacy classes. We have been so far a little bit cavalier matching the CFT partition function on the disk with the spacetime DBI action. The problem is that the divergences need to be regularized, and it is not *a priori* obvious that the regularization will not introduce an extra dependence on the location of the brane, which would spoil the relation between the CFT and the spacetime results for the D-brane action. The purpose of the discussion below is to show that this does not happen.

In the Euclidean AdS_3 the D-brane labeled by the principle continuous representation of SL(2,R) is a surface of constant $X^0 = \tilde{C}$, which is a two-sphere of radius $\sqrt{\tilde{C}^2 - 1}$. The DBI action (2.34) becomes finite, as the timelike coordinate takes a finite range $\tilde{t} \in [0, 2\pi]$. The situation with the D-brane located at constant $\tilde{X}^3 = C$ is more tricky, as it is an infinitely stretched H_2 plane (in the Euclidean case), with the DBI action (2.34) containing the divergent integral

$$\int dt\, dw\, \cosh w. \tag{4.47}$$

It is clear that this divergence is a manifestation of the fact that the H_2 plane has infinite volume. Let us now understand how this is reflected in the CFT analysis. Recall that the CFT partition sum on the disk is given by $U(h=0)$, which is related to $U(h=1)$ by the reflection symmetry. In the case of a D-brane at $X^0 = \tilde{C}$, we can use (3.74) to write

$$U(0) = -\frac{1}{\pi}\frac{\Gamma(1+u)}{\Gamma(1-u)} \int d^2y' \, \frac{U(1)}{(1+y'\bar{y}')^2} = -\frac{\Gamma(1+u)}{\Gamma(1-u)}\, U(1). \tag{4.48}$$

Hence, the partition sum $U(0)$ is indeed finite. Recall that the corresponding DBI action is also finite. Hence the description is free of volume divergences, and the CFT partition sum (4.24) is equal to the DBI action (2.22) up to a finite constant. Something interesting happens when we write the partition sum for a D-brane located at $\tilde{X}^3 = C$:

$$U(0) = -\frac{1}{\pi}\frac{\Gamma(1+u)}{\Gamma(1-u)} \int d^2y' \, \frac{U(1)}{(y'-\bar{y}')^2}. \tag{4.49}$$

This expression contains divergent integral, which, in light of the discussion above, must therefore signify the volume divergence. As we mentioned earlier, the result (3.27) may contain regularized infinities for the gluing conditions that are associated with D-branes located at $X^1, X^2, X^3 = \mathit{const}$.

The divergent integral that appears in (4.49) must be equal to (4.47)[1], up to a finite factor. To see that this factor is independent of C, it is useful to consider an overlap of the two boundary states $|\Theta_1\rangle$ and $|\Theta_2\rangle$. In the following we will use anti de Sitter coordinate system (2.31), in which these boundary states correspond to the D-branes located at $\psi = \psi_1$ and $\psi = \psi_2$, respectively. One way to compute the overlap is to insert the resolution of unity (4.3). This gives

$$\langle\Theta_1|\Theta_2\rangle = \int \frac{d^2y}{(y-\bar{y})^2} \int_0^\infty d\lambda\, A^2 \sin(i\Theta_1\lambda)\sin(i\Theta_2\lambda) = \frac{A^2\pi}{2}\,\delta(\psi_1-\psi_2)\int \frac{d^2y}{(y-\bar{y})^2}. \tag{4.50}$$

The same matrix element may be computed by using the coordinate representation of the boundary wavefunction

$$\langle g|\Theta_i\rangle = -A\pi \frac{\delta(\psi-\psi_i)}{\cosh\psi_i}. \tag{4.51}$$

The SL(2,R) invariant measure on H_3^+ is simply a volume form $[dg] = \cosh^2\psi \cosh w\, d\psi\, d\omega\, dt$, so the overlap becomes

$$\langle\Theta_1|\Theta_2\rangle = 2\pi\,\frac{A^2\pi}{2}\delta(\psi_1-\psi_2)\int dt\,dw\,\cosh w. \tag{4.52}$$

Hence we conclude that the divergent factors that appear in (2.34) and (4.43) match each other, up to a numerical factor, and do not introduce extra dependence on the location of the D-brane. Therefore the consistency of (2.34) and (4.43) is not an artifact of the choice of a coordinate system.

5 Discussion

In this paper we showed that the conformal bootstrap on the disk [1] gives boundary states that appear as surfaces of $X^i = const$ $(i = 0, ..., 3)$ in H_3^+. The gluing conditions determine the coordinate X^i that is normal to the D-brane worldvolume, while the complex parameter Θ that appears in the one-point function (3.72) determines the value of X^i where the D-brane is located. In H_3^+, the two-sphere at $X^0 = const$ corresponds to imaginary Θ, which is related via (3.78) to the continuous representation of SL(2,R). The hyperbolic plane at $X^a = const$ $(a = 1, 2, 3)$ corresponds to the Θ of the form (4.40). This value of Θ is consistent with the fusion of the degenerate operator $\Phi_{-\frac{1}{2}}$ to the boundary of the worldsheet and spacetime. The CFT partition sum on the disk for D-branes that we have studied reproduces the DBI action up to a finite normalization constant. Both

[1] One may introduce any suitable regularization, like restricting the range of coordinates in AdS_3, to define the divergent integrals.

quantities are infinite for D-branes that appear as hyperbolic planes, reflecting the infinite volume of the latter. However this volume infinity is shown to appear as a divergent integral over the y-plane in the CFT description, and therefore may be isolated and treated with the help of any suitable regulator.

A number of interesting things happen when one of the spacelike coordinates in H_3^+ is Wick rotated, giving Lorenzian AdS_3. Two-spheres at $X^0 = const$ turn into dS_2 surfaces. The X^0 coordinate becomes unrestricted, so one may now have dS_2 branes located at negative X^0. Of course, the D-brane at $X^0 = \tilde{C}$ and the one at $X^0 = -\tilde{C}$ are physically equivalent. In the worldsheet description the manifestation of this equivalence is that the one-point functions of these D-branes differ by a factor of $(-)^{2h}$, which does not affect conformal bootstrap. In AdS_3 it becomes possible to have D-branes located at $X^0 < 1$ (H_2 instantonic branes). These correspond to real Θ, which is related via (3.78) to the principal discrete representation of SL(2,R). Finally, D-branes at real $\tilde{X}^3 = const$ go into AdS_2 branes under the Wick rotation of X^1 or X^2.

An interesting question is the interpretation of the parameter Θ which labels AdS_2 branes, and is given by (4.40). One possibility may be that in the open string channel the expression (3.79) represents the sum of the principal continuous character and its spectral flow by one unit, but the precise identification needs more work[1]. Another important question that would be interesting to analyze is the geometric interpretation of boundary states that correspond to degenerate representations of SL(2,R). It is clear that corresponding D-branes must be localized in AdS_3, since the spectrum of open strings living on them contains a finite number of current algebra blocks. The possibility that they appear as two-spheres in H_3^+, mimicking the SU(2) case, seems to be ruled out by the identification of such two-spheres with extended D-branes that correspond to principal continuous series. Note that up to the normalization factor, which depends only on the half-integer h' that labels the boundary state, the one-point functions for the boundary states that correspond to the degenerate and principal discrete representations of SL(2,R) are equal. The semiclassical results for boundary wavefunctions are therefore indistinguishable. The meaning of this is not clear.

Let us make a brief comment about the role of the spectral flow in constructing D-branes in the H_3^+ WZNW model. Recall, that the ratio of structure constants (3.66). has in its

[1]The precise interpretation of (3.79) in the open string channel is difficult, since the (spectral flowed) character of the principal continuous representation is divergent.

right-hand side the square root of the ratio of two-point functions $\langle\Phi_h\Phi_h\rangle$ and $\langle\Phi_{\frac{k}{2}-h}\Phi_{\frac{k}{2}-h}\rangle$ [see also (3.51)] This ratio is essentially making up for the relative normalization of the operators Φ_h and $\Phi_{\frac{k}{2}-h}$. This leads us to the observation, that up to a phase,

$$\frac{U(\frac{k}{2}-h)}{\sqrt{\langle\Phi_{\frac{k}{2}-h}\Phi_{\frac{k}{2}-h}\rangle}} \sim \frac{U(h)}{\sqrt{\langle\Phi_h\Phi_h\rangle}}. \tag{5.1}$$

This implies that D-branes that correspond to degenerate representations of SL(2,R) are similar to D-branes in the SU(2) WZNW model. The relation (5.1) also hints that D-branes in the SL(2,R)/U(1) parafermion theory may be quite similar to their SU(2)/U(1) counterparts[2]. Understanding D-branes in SL(2,R)/U(1) is, of course, a very interesting open problem.

Acknowledgements

We gratefully acknowledge discussions with D. Kutasov. We would also like to thank D. Kutasov and A. Schwimmer for comments on the manuscript. This work was supported, in part, by DOE grant #DE-FG02-90ER40560.

[2]Recently D-branes in the coset model were studied from the geometric point of view [27].

Appendix

A The solutions of the Knizhnik-Zamolodchikov equation

In this appendix we write down the solutions of the Knizhnik-Zamolodchikov equation (3.7), which we reproduce below

$$\begin{aligned}\Big[&-z(z-1)(k-2)\partial_z+y(y-1)(z-y)\partial_y^2 \\ &+\Big[(\Delta+1)(-y^2+2zy-z)-2h_0y(y-1)-2yh_1(z-1)-2(y-1)h_2z\Big]\partial_y \\ &+[2h_0\Delta(z-y)-2h_0h_1(z-1)-2h_0h_2z]\Big]H\left(\begin{array}{cc}h_0 & h_1\\ h_2 & h_3\end{array},y,z\right)=0\end{aligned} \tag{A.1}$$

for the special cases that appear in the main text. We first consider the conformal blocks that appear in the two-point function, which can be written in the s and t channels as

$$\langle\Phi_{-\frac{1}{2}}(y_1,z_1)\Phi_h(y_2,z_2)\rangle=\frac{|y_2-\bar{y}_2|^{-1-2h}}{|y_1-\bar{y}_2|^{-2}}\,\frac{|z_2-\bar{z}_2|^{-\frac{3u}{2}-2\Delta_h}}{|z_1-\bar{z}_2|^{-3u}}\,H\left(\begin{array}{cc}-1/2 & h\\ -1/2 & h\end{array},y,z\right) \tag{A.2}$$

and

$$\langle\Phi_{-\frac{1}{2}}(y_1,z_1)\Phi_h(y_2,z_2)\rangle=\frac{|y_2-\bar{y}_2|^{-1-2h}}{|y_1-\bar{y}_2|^{-2}}\,\frac{|z_2-\bar{z}_2|^{-\frac{3u}{2}-2\Delta_h}}{|z_1-\bar{z}_2|^{-3u}}\,H\left(\begin{array}{cc}-1/2 & -1/2\\ h & h\end{array},1-y,1-z\right) \tag{A.3}$$

respectively, where y and z are the cross-ratios defined in the main text. The conformal blocks of this sort have been previously known [22, 21, 19], so we present the results here for completeness.

Consider $h_0=-\frac{1}{2}$. It is not hard to see that

$$H\left(\begin{array}{cc}-1/2 & h_1\\ h_2 & h_3\end{array},y,z\right)=C_1[yH_1^+(z)+H_0^+(z)]+C_2[yH_1^-(z)+H_0^-(z)], \tag{A.4}$$

where C_1 and C_2 are some constants. One can convince oneself that this is the case by substituting the expression (A.4) into (A.1) and noting that the $\mathcal{O}(y^2)$ term is trivially zero. The vanishing of $\mathcal{O}(y)$ and $\mathcal{O}(1)$ terms implies that $H_1^\pm(z)$ and $H_0^\pm(z)$ satisfy

$$\left[-z(z-1)(k-2)\partial_z+\left(\left(\frac{3}{2}-h_3\right)z+h_1-1\right)\right]H_1^\pm+\Delta H_0^\pm=0, \tag{A.5}$$

$$-z(z-1)(k-2)\partial_zH_0(z)+\left[h_2-h_1+h_3-\frac{1}{2}\right]H_1^\pm+\left[\left(h_3+\frac{1}{2}\right)z-h_1\right]H_0^\pm=0. \tag{A.6}$$

Substituting the first equation into the second one obtains the second-order differential equation for $H_1^{\pm}(z)$, which has two linearly independent solutions. (This explains the need for the superscript $\pm$.) After the substitution

$$H_1^{\pm}(z) = z^{\beta_1^{\pm}}(1-z)^{\beta_2^{\pm}} K^{\pm}(z), \tag{A.7}$$

$$\beta_i^{+} = u(1-h_i), \qquad \beta_i^{-} = uh_i. \tag{A.8}$$

One has the hypergeometric equation for $K^{\pm}(z)$

$$\Big[z(1-z)\partial_z^2 + (C-(A+B+1)z)\partial_z - AB\Big] K^{\pm}(z) = 0, \tag{A.9}$$

where

$$\begin{aligned} A+B+1 &= 2\left(\beta_1+\beta_2+\frac{k-3}{k-2}\right), \\ AB &= (\beta_1+\beta_2)\left(\beta_1+\beta_2-1+2\frac{k-3}{k-2}\right) + \frac{\left(\frac{3}{2}-h_3\right)}{k-2}\left(\frac{h_3+\frac{1}{2}}{k-2}-1\right), \\ C &= 2\beta_1+\frac{k-3}{k-2}. \end{aligned} \tag{A.10}$$

The solution of (A.9) that is regular at $z=0$ is a hypergeometric function $F(A,B,C,z)$ with A, B, C given by the solution of (A.10). Using (A.5), (A.7), (A.8) and (A.10), together with the identification $h_1 = h_3 = h$, $h_2 = -1/2$, $\Delta = -1$, one may reproduce the conformal blocks (3.13)-(3.16). The alternative identification $h_2 = h_3 = h$, $h_1 = -1/2$, gives rise to

$$H\begin{pmatrix} -1/2 & -1/2 & ,1-y,1-z \\ h & h & \end{pmatrix} = \tilde{C}_1(1-z)^{\frac{3u}{2}} z^{u(1-h)} F(u, 1+2u(1-h), 1+2u, 1-z)(1-y) + \cdots, \tag{A.11}$$

where $\tilde{C}_1$ is some constant, and the dots stand for conformal blocks whose explicit form will not be needed to us.

The equation (A.1) can also be solved for the case $h_0 = -\frac{k+2}{2}$ [22, 21]. The solution involves hypergeometric functions of two arguments. We will instead look at the conformal block with $h_3 = \frac{k+1}{2}$. It was shown in [19] that the relevant solution takes a simple form:

$$\begin{aligned} H\begin{pmatrix} h_0 & h_1 & ,y,z \\ h_2 & (k+1)/2 & \end{pmatrix} &= C_1'[(y-z)^{-\Delta} H_0'^{+}(z) + (y-z)^{-\Delta-1} H_1'^{+}(z)] \\ &\quad + C_2'[(y-z)^{-\Delta} H_0'^{-}(z) + (y-z)^{-\Delta-1} H_1'^{-}(z)]. \end{aligned} \tag{A.12}$$

Again, this can be easily seen by substituting (A.12) into (A.1) and observing that the $\mathcal{O}(-\Delta+1)$ and $\mathcal{O}(-\Delta-2)$ terms vanish trivially. The requirement for the vanishing of $\mathcal{O}(-\Delta)$ and $\mathcal{O}(-\Delta-1)$ terms leads to the system of two differential equations similar to (A.5)–(A.6):

$$[z(z-1)(k-2)\partial_z + a]\, H_0'^{\pm}(z) + [\Delta + 1 - 2h_0]\, H_1'^{\pm}(z) = 0, \tag{A.13}$$
$$[z(z-1)(k-2)\partial_z + b]\, H_1'^{\pm}(z) - \Delta z(z-1) H_0'^{\pm}(z) = 0, \tag{A.14}$$

where

$$a = z\,[\Delta(2h_2 + 1 - k - 2\Delta) + 2h_0(h_1 + h_2)] + \Delta(\Delta - 2h_2 + k) - 2h_0 h_1, \tag{A.15}$$
$$b = z\,[(\Delta+1)(3-2h_0-k)+2h_0(h_1+h_2)] - (\Delta+1)(2h_2+1-k-\Delta) - 2h_0 h_1. \tag{A.16}$$

Consider the two-point function dual to (A.2)

$$\begin{aligned}\langle \Phi_h(y_1,z_1)\Phi_{\frac{k+1}{2}}(y_2,z_2)\rangle &= |y_2-\bar{y}_2|^{-k-1+2h}|y_1-\bar{y}_2|^{-4h} \\ &\quad |z_2-\bar{z}_2|^{2\Delta_h - 2\Delta_{\frac{k+1}{2}}}|z_1-\bar{z}_2|^{-4\Delta_h} H\left(\begin{array}{cc} h & (k+1)/2 \\ h & (k+1)/2 \end{array}, y, z\right).\end{aligned} \tag{A.17}$$

The analogs of (A.7)-(A.8) for the conformal blocks that appear in the expansion (A.12) of $H\left(\begin{array}{cc} h & (k+1)/2 \\ h & (k+1)/2 \end{array}, y, z\right)$ are

$$H_1'^{\pm}(z) = z^{\beta_1^{\pm}}(1-z)^{\beta_2^{\pm}} K'^{\pm}(z), \tag{A.18}$$
$$\beta_1^{+} = uh(k-1), \qquad \beta_2^{+} = 2uh(h-1), \tag{A.19}$$
$$\beta_1^{-} = 1 + h + u(1-h), \qquad \beta_2^{-} = 2uh(h-1).$$

Solving (A.13)–(A.14) gives (3.58)–(3.61). We will also need the conformal block that appears in the boundary expansion of the two-point function (A.17) [This expression is dual to (A.3)]

$$\begin{aligned}\langle \Phi_h(y_1,z_1)\Phi_{\frac{k+1}{2}}(y_2,z_2)\rangle &= |y_2-\bar{y}_2|^{-k-1+2h}|y_1-\bar{y}_2|^{-4h} \\ &\quad |z_2-\bar{z}_2|^{2\Delta_h - 2\Delta_{\frac{k+1}{2}}}|z_1-\bar{z}_2|^{-4\Delta_h} H\left(\begin{array}{cc} h & h \\ (k+1)/2 & (k+1)/2 \end{array}, 1-y, 1-z\right).\end{aligned} \tag{A.20}$$

The relevant solution is

$$\begin{aligned}H\left(\begin{array}{cc} h & h \\ (k+1)/2 & (k+1)/2 \end{array}, 1-y, 1-z\right) &= \\ &\tilde{C}_1'\,([1-x]-[1-z)])^{-2h}\,(1-z)^{2uh(h-1)} z^{h+uh} F(u, 2uh, 1+2u, 1-z) + \cdots .\end{aligned} \tag{A.21}$$

B Spectral flow and Parafermionic Construction

In this appendix we show how three and four point correlation functions of the primaries of the $SL(2,R)$ WZNW model transform under the spectral flow by one unit. As it was shown in [23] the spectral flow by one unit ($w=1$) maps the highest weight representation into lowest weight representation and lowest weight($w=-1$) into highest weight:

$$\begin{aligned} \Phi^{hw}_{h,-h} &\to \Phi^{lw}_{\frac{k}{2}-h,\frac{k}{2}-h}, \\ \Phi^{lw}_{h,h} &\to \Phi^{hw}_{\frac{k}{2}-h,-\frac{k}{2}+h}. \end{aligned} \tag{B.1}$$

For simplicity we are suppressing the antiholomorphic part of the operators. The operators of the current algebra corresponding to the highest weight representation can be expressed in terms of free bosonic field (with the wrong sign in the propagator in the contrast to the $SU(2)$ case) and so called parafermionic operators.

$$\Phi^{hw}_{h;m} = e^{im\sqrt{\frac{2}{k}}\phi(z)}\Psi^{hw}_{h;m}; \quad m \in -h+Z. \tag{B.2}$$

where $\Phi^{hw}_{h;m}$ are defined as follows

$$\begin{aligned} \Phi^{hw}_{h;m}(z) &\equiv (J^-_0)^{-h-m}\Phi^{hw}_{h;-h}(z) \quad m<-h, \\ \Phi^{hw}_{h;m}(z) &\equiv (J^+_{-1})^{m+h}\Phi^{hw}_{h;-h}(z) \quad m\geq -h. \end{aligned} \tag{B.3}$$

Note that the highest weight field $\Phi^{hw}(y,z)$ can be written in terms of $\Phi^{hw}_{h;m}(z)$ in the following way

$$\Phi^{hw}(y,z) = \sum_{m=-\infty}^{-h} \Phi^{hw}_{h;m} y^{-h-m}(z). \tag{B.4}$$

Now using (B.3) one can easily derive the conformal weights for parafermions

$$\Delta_{\Psi^{hw}_{h,m}} = \left\{ \begin{array}{l} -\frac{h(h-1)}{k-2} + \frac{m^2}{k}; \quad m\leq -h \\ -\frac{h(h-1)}{k-2} + \frac{m^2}{k} + (m+h); \quad m\geq -h \end{array} \right\}. \tag{B.5}$$

Similarly for the lowest weight representation we have the lowest weight parafermions which have the following conformal dimensions

$$\Delta_{\Psi^{lw}_{h,m}} = \left\{ \begin{array}{l} -\frac{h(h-1)}{k-2} + \frac{m^2}{k}; \quad m\geq h \\ -\frac{h(h-1)}{k-2} + \frac{m^2}{k} + (h-m); \quad m\leq h \end{array} \right\}. \tag{B.6}$$

The parafermionic theory is believed to be unitary (see for example [28]). From the expressions (B.5), (B.6) we see that the operators $\Psi^{lw}_{\frac{k}{2};\frac{k}{2}}$ and $\Psi^{hw}_{\frac{k}{2};-\frac{k}{2}}$ have zero conformal weight and should be proportional to the identity operator. It is also natural to assume that $\Psi^{hw}_{h;m}(z)$ and $\Psi^{lw}_{\frac{k}{2}-h;m+\frac{k}{2}}(z)$ are identified up to a multiplicative constant (which in general depends on (h,m) and is related to the normalization of the operators) since they have the same worldsheet dimension. We will need in particular relation between $\Psi^{hw}_{h;-h}(z)$ and its counterpart under the spectral flow

$$\Psi^{hw}_{h;-h}(z) = B(h)\Psi^{lw}_{\frac{k}{2}-h;\frac{k}{2}-h}(z). \tag{B.7}$$

The relative normalization $B(h)$ of parafermions adopted in this paper is

$$B(h) = \sqrt{\frac{\langle\Phi_h\Phi_h\rangle}{\langle\Phi_{\frac{k}{2}-h}\Phi_{\frac{k}{2}-h}\rangle}}, \tag{B.8}$$

where $\langle\Phi_h\Phi_h\rangle$ is the two point function of the operator Φ_h stripped of the worldsheet and spacetime coordinates dependence.

This assumption will allow us to derive the transformation properties of three and four point functions under spectral flow. One can use the conformal symmetry (spacetime and worldsheet) to write a three point correlation function in the following form

$$\langle\Phi_{h_1}(y_1,z_1)\Phi_{h_2}(y_2,z_2)\Phi_{h_3}(y_3,z_3)\rangle = D(h_1,h_2,h_3)\prod_{i<j}^{3} y^{\lambda_{ij}}z^{\Delta_{ij}}, \tag{B.9}$$

where $y_{ij} = y_i - y_j$ and

$$\lambda_{12} = h_1 + h_2 - h_3, \tag{B.10}$$

$$\lambda_{13} = h_1 + h_3 - h_2, \tag{B.11}$$

$$\lambda_{23} = h_2 + h_3 - h_1. \tag{B.12}$$

z_{ij} and Δ_{ij} are defined in the similar manner with $h_i \to \Delta_i$. The coefficient $D(h_1,h_2,h_3)$ is symmetric under permutation of its arguments since correlation function should not depend on the order of operators and can be easily extracted from (B.9)

$$D(h_1,h_2,h_3) = \lim_{\substack{y_3\to\infty \\ z_3\to\infty}} \langle\Phi_{h_1}(0,0)\Phi_{h_2}(1,1)\Phi_{h_3}(y_3,z_3)\rangle y_3^{2h_3}z_3^{2\Delta_3}. \tag{B.13}$$

So we conclude that in order to compute $D(h_1,h_2,h_3)$ one needs to know correlators of the form

$$\langle\Phi_{h_1}(0,0)|\Phi_{h_2}(1,1)|\Phi_{h_3}(\infty,\infty)\rangle = \langle\Phi_{h_1;-h_1}(0)|\Phi_{h_2}(1,1)|\Phi_{h_3;h_3}(\infty)\rangle. \tag{B.14}$$

Expanding the r.h.s. of the above equation in spacetime coordinate and using parafermion representation we will get

$$\langle\Phi_{h_1;-h_1}(0)|\Phi_{h_2}(1,1)|\Phi_{h_3;h_3}(\infty)\rangle = \sum_{m_2}\langle\Phi_{h_1;-h_1}(0)|\Phi_{h_2;m_2}(1)|\Phi_{h_3;h_3}(\infty)\rangle =$$

$$\sum_{m_2}\langle\Psi_{h_1;-h_1}(0)|\Psi_{h_2;m_2}(1)|\Psi_{h_3;h_3}(\infty)\rangle\delta(m_2+h_3-h_1)=$$

$$\sum_{m_2}B(h_1)B(h_3)\langle\Psi_{\frac{k}{2}-h_1;\frac{k}{2}-h_1}(0)|\Psi_{h_2;m_2}(1)|\Psi_{\frac{k}{2}-h_3;-\frac{k}{2}+h_3}(\infty)\rangle\delta(m_2+(-\frac{k}{2}+h_3)+(\frac{k}{2}-h_1))= \tag{B.15}$$

$$B(h_1)B(h_3)\langle\Phi_{\frac{k}{2}-h_1;\frac{k}{2}-h_1}(0)|\Phi_{h_2}(1,1)|\Phi_{\frac{k}{2}-h_3;-\frac{k}{2}+h_3}(\infty)\rangle.$$

In the derivation above we used the free field three point function which is just a delta function. One can convince oneself that the last line of (B.15) is equal to $B(h_1)B(h_3)D(k/2-h_1,h_2,k/2-h_3)$, so we obtained the following relation

$$D(h_1,h_2,h_3)=B(h_1)B(h_3)D(k/2-h_1,h_2,k/2-h_3). \tag{B.16}$$

Using the explicit expression for $D(h_1,h_2,h_3)$ from [22] one can check that (B.16) indeed holds.

Let us now find the transformation of four point correlation function under the spectral flow. For the rest of this appendix we will set $B(h)$ to one to simplify formulae (one can easily restore the overall normalizations guided by the example of three point function). Recall that using the conformal symmetry (both worldsheet and spacetime)(cr. (3.1)) the four point function can be written as

$$\langle\Phi_{h_1}(y_1,z_1)\Phi_{h_0}(y_0,z_0)\Phi_{h_2}(y_2,z_2)\Phi_{h_3}(y_3,z_3)\rangle=\prod y_{ij}^{\mu_{ij}}z_{ij}^{\nu_{ij}}H\left(\begin{array}{cc}h_0&h_1\\h_2&h_3\end{array},y,z\right), \tag{B.17}$$

where H is defined as

$$H\left(\begin{array}{cc}h_0&h_1\\h_2&h_3\end{array},y,z\right)=\lim_{\substack{y_3\to\infty\\z_3\to\infty}}\langle\Phi_{h_1}(0,0)\Phi_{h_0}(y,z)\Phi_{h_2}(1,1)\Phi_{h_3}(y_3,z_3)\rangle y_3^{2h_3}z_3^{\Delta_3}. \tag{B.18}$$

We see that in order to compute H one needs to know correlators of the form

$$\langle\Phi_{h_1}(0,0)|\Phi_{h_0}(y,z)\Phi_{h_2}(1,1)|\Phi_{h_3}(\infty,\infty)\rangle=\langle\Phi_{h_1;-h_1}(0)|\Phi_{h_0}(y,z)\Phi_{h_2}(1,1)|\Phi_{h_3;h_3}(\infty)\rangle. \tag{B.19}$$

We will also need the following relation

$$\langle\Phi_{h_1}(\infty,0)|\Phi_{h_0}(y,z)\Phi_{h_2}(1,1)|\Phi_{h_3}(0,\infty)\rangle=y^{-2h_0}H\left(\begin{array}{cc}h_0&h_1\\h_2&h_3\end{array},\frac{1}{y},z\right)=$$

$$\langle\Phi_{h_1;h_1}(0)|\Phi_{h_0}(y,z)\Phi_{h_2}(1,1)|\Phi_{h_3;-h_3}(\infty)\rangle, \tag{B.20}$$

To prove (B.20) let us rewrite (B.17) with $y_1 \leftrightarrow y_3$

$$\langle \Phi_{h_1}(y_3, z_1)\Phi_{h_0}(y_0, z_0)\Phi_{h_2}(y_2, z_2)\Phi_{h_3}(y_1, z_3)\rangle = \prod y_{ij}^{\tilde{\mu}_{ij}} \prod z_{ij}^{\nu_{ij}} H \left(\begin{array}{cc} h_0 & h_1 \\ h_2 & h_3 \end{array}, \tilde{y}, z \right), \quad \text{(B.21)}$$

where nonzero $\tilde{\mu}_{ij}$

$$\tilde{\mu}_{01} = -2h_0, \quad \text{(B.22)}$$

$$\tilde{\mu}_{13} = -h_1 - h_3 + h_0 + h_2, \quad \text{(B.23)}$$

$$\tilde{\mu}_{12} = -h_2 - h_3 + h_0 + h_1, \quad \text{(B.24)}$$

$$\tilde{\mu}_{23} = -h_0 - h_1 - h_2 + h_3, \quad \text{(B.25)}$$

are obtained from μ_{ij} simply by exchange of indices $1 \leftrightarrow 3$ and $\tilde{y} = \frac{1}{y}$. Taking the proper limit in (B.21) we arrive to the expression (B.20).

Now, using (B.3) and expanding the operators in spacetime coordinate, we have

$$\begin{aligned} &\sum_{m_0,m_2} \langle \Phi_{h_1;-h_1}(0)|\Phi_{h_0;m_0}(z)\Phi_{h_2;m_2}(1)|\Phi_{h_3;h_3}(\infty)\rangle y^{-h_0-m_0} = \\ &\sum_{m_0,m_2} \langle \Psi_{h_1;-h_1}(0)|\Psi_{h_0;m_0}(z)\Psi_{h_2;m_2}(1)|\Psi_{h_3;h_3}(\infty)\rangle y^{-h_0-m_0} \times \\ &\langle e^{i\sqrt{\frac{2}{k}}(-h_1)\phi(0)}|e^{i\sqrt{\frac{2}{k}}(m_0)\phi(z)}e^{i\sqrt{\frac{2}{k}}(m_2)\phi(1)}|e^{i\sqrt{\frac{2}{k}}(h_3)\phi(\infty)}\rangle = \\ &\sum_{m_0,m_2} \langle \Psi_{h_1;-h_1}(0)|\Psi_{h_0;m_0}(z)\Psi_{h_2;m_2}(1)|\Psi_{h_3;h_3}(\infty)\rangle y^{-h_0-m_0} z^{m_0 h_1 \frac{2}{k}}(1-z)^{-\frac{2}{k}m_0 m_2}. \end{aligned} \quad \text{(B.26)}$$

In the second line we used the representation of the operators of $SL(2,R)$ in terms of free field and parafermions, and in the last line the four point function for operators in the free field theory was used.

After performing the spectral flow to the "in" and "out" states we will obtain

$$\begin{aligned} &\langle \Phi_{\frac{k}{2}-h_1}(\infty, 0)|\Phi_{h_0}(y,z)\Phi_{h_2}(1,1)|\Phi_{\frac{k}{2}-h_3}(0,\infty)\rangle = \\ &\sum_{m_0,m_2} \langle \Phi_{\frac{k}{2}-h_1;\frac{k}{2}-h_1}(0)|\Phi_{h_0;m_0}(z)\Phi_{h_2;m_2}(1)|\Phi_{\frac{k}{2}-h_3;-\frac{k}{2}+h_3}(\infty)\rangle y^{-h_0-m_0} = \\ &\sum_{m_0,m_2} \langle \Psi_{h_1;-h_1}(0)|\Psi_{h_0;m_0}(z)\Psi_{h_2;m_2}(1)|\Psi_{h_3;h_3}(\infty)\rangle y^{-h_0-m_0} z^{-\frac{2}{k}(\frac{k}{2}-h_1)m_0}(1-z)^{-\frac{2}{k}m_0 m_2} = \\ &\langle \Phi_{h_1}(0,0)|\Phi_{h_0}(yz,z)\Phi_{h_2}(1,1)|\Phi_{h_3}(\infty,\infty)\rangle z^{h_0}. \end{aligned} \quad \text{(B.27)}$$

Using (B.20), (B.27) can be rewritten in the following form:

$$y^{h_0} H \left(\begin{array}{cc} h_0 & \frac{k}{2}-h_1 \\ h_2 & \frac{k}{2}-h_3 \end{array}, y, z \right) = \left(\frac{z}{y}\right)^{h_0} H \left(\begin{array}{cc} h_0 & h_1 \\ h_2 & h_3 \end{array}, \frac{z}{y}, z \right). \quad \text{(B.28)}$$

In order to check the relation (B.28) we recall that in the Appendix A (see also (3.58)-(3.61)) we computed

$$H\left(\begin{array}{cc} -1/2 & h \\ -1/2 & h \end{array}, y, z\right) = H_0(z) + yH_1(z), \tag{B.29}$$

$$H\left(\begin{array}{cc} h & k/2+1/2 \\ h & k/2+1/2 \end{array}, y, z\right) = (y-z)^{-2h} H_0'(z) + (y-z)^{-2h-1} H_1'(z). \tag{B.30}$$

In order to compare these two expressions we should make some transformations. By changing the positions of operators in (B.17) and demanding that the result do not change under this transformation we have

$$H\left(\begin{array}{cc} h & -1/2 \\ h & -1/2 \end{array}, y, z\right) = (1-y)^{-2h-1}(1-z)^{-2\Delta_h+2\Delta_{-1/2}} H\left(\begin{array}{cc} -1/2 & h \\ -1/2 & h \end{array}, y, z\right). \tag{B.31}$$

Substituting (B.31) into (B.28) we will get

$$\begin{aligned} H_0'(z) &= z^h (1-z)^{-2\Delta_h+2\Delta_{-1/2}} H_0(z), \\ H_1'(z) &= z^{h+1} (1-z)^{-2\Delta_h+2\Delta_{-1/2}} (H_0(z) + H_1(z)). \end{aligned} \tag{B.32}$$

A straightforward comparison of (3.14), (3.16) with (3.58), (3.60) verifies the first line in the above equation. The verification of the second line is technically more involved and one needs to employ Gauss' recursion formulas for hypergeometric functions. Using (D.1), (3.15), (3.16), (3.61) we see that

$$H_1'^{-}(z) = z^{h+1}(1-z)^{-2\Delta_h+2\Delta_{-1/2}}(H_0^-(z) + H_1^-(z)) \tag{B.33}$$

holds. Finally we can combine (D.4),(3.13),(3.14),(3.59) to show that

$$H_1'^{+}(z) = z^{h+1}(1-z)^{-2\Delta_h+2\Delta_{-1/2}}(H_0^+(z) + H_1^+(z)). \tag{B.34}$$

This concludes our check.

C Some useful integrals

In this Appendix we compute some integrals that were important for the discussion in section 3. The integrals of this sort were frequently encountered in section 4, which we will heavily borrow from. We start by computing the integral that appears in (3.74).

$$\int d^2y' \, \frac{|y-y'|^{-4h}}{(1+y'\bar{y}')^{2-2h}} = \int d^2y' \left(y_1'^2 + y_2'^2 - 2Ry_1' + R^2\right)^{-2h} \left(1 + y_1'^2 + y_2'^2\right)^{2h-2}, \tag{C.1}$$

where we introduced $R = |y|$. Exponentiating the integrand, completing the square, and performing the integral over the y'-plane one obtains

$$\frac{\pi}{\Gamma(2h)\Gamma(2-2h)} \int_0^\infty \frac{t^{2h-1}s^{1-2h}}{t+s} \exp\left(\frac{R^2t^2}{t+s} - tR^2 - s\right). \tag{C.2}$$

Writing $s = \alpha t$, as usual, and integrating over t leaves us with

$$\frac{\pi}{\Gamma(2h)\Gamma(2-2h)} \int_0^\infty d\alpha \frac{\alpha^{-2h}}{\alpha + (1+R^2)}, \tag{C.3}$$

which depends only on $(1 + y\bar{y})$, as it should. After rescaling the integral above becomes a simple Veneziano integral which is not hard to do. The result is

$$\frac{\pi}{1-2h} \frac{1}{(1+y\bar{y})^{2h}}. \tag{C.4}$$

Substituting this into (3.74) we recover the advertised result

$$U(h) = -\mathcal{R}(h)U(1-h). \tag{C.5}$$

This result is important for fixing the solution of the bootstrap equation to have the form (3.72)

Let us now compute the h-dependent factor that appears in the analog of (3.75) for the partition sum of an open string stretched between a D-brane described by the one-point function of the form $\langle \Phi_h(y,\bar{y}) \rangle = \frac{U(h)}{y-\bar{y})^{2h}}$ and the basic D-instanton. This factor is given by the following integral

$$\int d^2y \, (y-\bar{y})^{2-2h} \, (1+y\bar{y})^{-2h}. \tag{C.6}$$

In fact, this integral may be computed in two ways. First let us notice that the expression above can be represented by an integral of the form (4.9) with $X^0 = X^1 = X^2 = 0$ and $X^3 = 1$. In the parameterization used in section 4 this corresponds to $\tilde{\psi} = i\frac{\pi}{2}$. The value of this integral can be read from (4.17)

$$\frac{\pi}{2h-1} \frac{\sinh[(2h-1)\tilde{\psi}]}{\sinh \tilde{\psi}} = \frac{\pi}{2h-1} \sinh[i(2h-1)\frac{\pi}{2}]. \tag{C.7}$$

The alternative computation uses the integral that appears in (4.32) with the identification $X^0 = 1$, $X^1 = X^2 = X^3 = 0$. This corresponds to $\psi = 0$, and using (4.38) we recover (C.7). Substituting (C.7) into (3.75) gives (3.79).

D Some properties of hypergeometric functions

In this appendix we give some formulas that are used in this paper. We will need the following Gauss recursion formulas for hypergeometric functions [26]

$$cF(a,b,c;z)+(b-c)F(a+1,b,c+1;z)-b(1-z)F(a+1,b+1,c+1;z)=0, \quad (D.1)$$

$$cF(a,b,c;z)-cF(a+1,b,c;z)+bzF(a+1,b+1,c+1;z)=0, \quad (D.2)$$

$$cF(a,b,c;z)-(c-b)F(a,b,c+1;z)-bF(a,b+1,c+1;z)=0. \quad (D.3)$$

In Appendix B we also needed the following relation that hypergeometric functions satisfy

$$F(a,b,b-a;z)+\frac{a}{b-a}zF(a+1,b,b-a+1;z)-(1-z)F(a+1,b,b-a;z)=0, \quad (D.4)$$

which we prove here.

$$\begin{aligned}&F(a,b,b-a;z)+\tfrac{a}{b-a}zF(a+1,b,b-a+1;z)-(1-z)F(a+1,b,b-a;z)=\\&z\left[F(a+1,b,b-a;z)+\tfrac{a}{b-a}F(a+1,b,b-a+1;z)-\tfrac{b}{b-a}F(a+1,b+1,b-a+1;z)\right]=\\&\tfrac{z}{b-a'+1}[(b-a'+1)F(a',b,b-a+1;z)-bF(a',b+1,b-a+2;z)-\\&(b-a'+1-b)F(a',b,b-a'+2;z)]=0,\end{aligned} \quad (D.5)$$

where $a'=a+1$. To get the second line we used (D.3) and the last equality follows from (D.3).

We are also giving the transformation properties of hypergeometric functions. Under $z\to 1/z$:

$$\begin{aligned}F(a,b,c;z)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-\frac{1}{z})^aF(a,a+1-c;a+1-b;\frac{1}{z})+\\&\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-\frac{1}{z})^bF(b,b+1-c;b+1-a;\frac{1}{z})\ .\end{aligned} \quad (D.6)$$

Under $z\to 1-z$:

$$\begin{aligned}F(a,b,c;1-z)=&\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b+1-c;z)+\\&z^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c+1-a-b;z)\ .\end{aligned} \quad (D.7)$$

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